Front propagation in infinite cylinders: a variational approach

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Abstract

We present a comprehensive study of front propagation for scalar reaction-diffusion-advection equations in infinite cylinders in the presence of transverse advection by a potential flow and mixtures of Dirichlet and Neumann boundary conditions. We take on a variational point of view, based on the fact that the considered equation is a gradient flow in an exponentially weighted L^2-space generated by a certain functional, when the dynamics is considered in the reference frame moving with constant velocity along the cylinder axis. In particular, certain traveling wave solutions in the form of fronts connecting different equilibria are critical points of this functional. Under very general assumptions, we prove existence, uniqueness, monotonicity, asymptotic behavior at infinity of the special traveling wave solutions which are minimizers of the considered functional. We also prove that if the functional does not have non-trivial minimizers, there is a traveling wave solution characterized by a certain "minimal speed". In all cases, the speeds of these waves determine the asymptotic propagation speed of the solutions of the initial-value problem for a large class of initial data that decay sufficiently rapidly exponentially in the direction of propagation. We also perform a detailed variational study of the limit problem arising in the context of combustion theory that leads to a free boundary problem and derive sharp upper and lower bounds for the propagation velocity, as well as establishing convergence of the regularizing approximations to the solution of the free boundary problem. The conclusions of the analysis are illustrated by a number of numerical examples. This study generalizes and extends the existing theory of propagation phenomena in reaction-diffusion equations which is based largely on the applications of the Maximum Principle.

Contents

1 Introduction

This paper is concerned with the study of front propagation in reaction-diffusion-advection problems in cylinders which arise in numerous applications and, specifically, in the context of combustion modeling. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain (not necessarily simply connected), and consider $\Sigma = \Omega \times \mathbb{R}$, an infinite cylinder in $\mathbb{R}^n$. In Σ, we shall consider the following parabolic equation

$$u_t + \mathbf{v} \cdot \nabla u = \Delta u + f(u, y). \tag{1.1}$$

Here $u = u(x,t) \in \mathbb{R}$ is the dependent variable (corresponding, e.g., to temperature in combustion problems), $\mathbf{v} = \mathbf{v}(y) \in \mathbb{R}^n$ is an imposed advective flow, and $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is a nonlinear reaction term. By $x = (y, z) \in \Sigma$, we denote a point with coordinate $y \in \Omega$ on the cylinder cross-section and $z \in \mathbb{R}$ along the cylinder axis. We also assume that

$$u = 0 \tag{1.2}$$

is a trivial solution of (1.1).

We are interested in a particular situation in which the flow $\mathbf{v}$ is *transverse* to the axis of the cylinder, i.e., when $\mathbf{v}$ does not have a component along z. Furthermore, we assume that the flow $\mathbf{v}(y)$ is potential:

$$\mathbf{v} = (-\nabla_y \varphi, 0), \qquad \varphi : \overline{\Omega} \to \mathbb{R}. \tag{1.3}$$

Respectively, those parts of $\partial\Omega$, denoted by $\partial\Omega_+$, on which $\nu \cdot \nabla_y \varphi > 0$ are the inlets and those, denoted by $\partial\Omega_-$, where $\nu \cdot \nabla_y \varphi < 0$ are the outlets (of fuel in combustion problems, e.g.). Here and below ν is the outward normal to $\partial\Omega$

(or $\partial\Sigma$). We denote those parts of $\partial\Omega$ on which $\nu \cdot \nabla_y \varphi = 0$ by $\partial\Omega_0$, these are impermeable walls. Consistently with this interpretation of the flow $\mathbf{v}$, we impose the following boundary conditions:

$$u\big|_{\partial\Sigma_\pm} = 0, \qquad \nu \cdot \nabla u\big|_{\partial\Sigma_0} = 0. \tag{1.4}$$

on $\partial\Sigma_\pm = \partial\Omega_\pm \times \mathbb{R}$ and $\partial\Sigma_0 = \partial\Omega_0 \times \mathbb{R}$. We demonstrate in more detail how this mathematical setup arises in modeling combustion phenomena in Sec. 2. Naturally, the dimension of physical interest is $n = 3$. Also, a problem on a two-dimensional strip is of special physical importance (our numerical examples will be from this category). Of course, the case of a purely reaction-diffusion equation ($\varphi = 0$) with either Dirichlet or Neumann boundary conditions is included in our formulation. In fact, it is the diffusion part in combination with Dirichlet boundary conditions and/or inhomogeneous reaction term that present the main difficulties in the analysis of this problem; nevertheless, for the sake of generality and because of the importance to applications we will treat the case of a general transverse potential flow here.

Note that we assumed that the nonlinearity does not depend explicitly on z, implying translational invariance along the z-direction. This gives rise to a possibility of existence of traveling wave solutions $u(x,t) = \bar{u}(y, z - ct)$ of (1.1). Moreover, a particular class of the traveling wave solutions in which the $u = 0$ equilibrium is *invaded*, i.e. fronts propagating with speed $c > 0$ from left to right, is of special interest in the context of the initial value problem governed by (1.1) [24, 60, 62]. These solutions and the associated propagation phenomena governed by (1.1) will be the main subject of the present paper.

Equation (1.1) has been the subject of great many studies, dating back to the pioneering work of Fisher [26] and Kolmogorov, Petrovsky and Piskunov [35]. While the results for the scalar reaction-diffusion equation on the real line are by now classic (see, e.g., reviews [4, 24, 60]), the analysis of higher-dimensional problems in unbounded domains is more recent [5, 6, 8, 12, 27, 40, 49, 62]. These problems naturally arise in the study of flame propagation in combustion theory [16, 64], which also motivates the introduction of advection by a flow. In this context the studies of traveling wave solutions and propagation phenomena for a number of different types of nonlinearities goes back to the work of Berestycki, Larrouturou, and Lions [10], which are summarized and extended by Berestycki and Nirenberg in [12] and by Roquejoffre in [49]. This work concentrates on the analysis of existence and properties of the front-like traveling wave solutions in cylinders with Neumann boundary conditions in the presence of shear flow along the cylinder axis. Note that this situation is quite different from the case of a potential flow perpendicular to the cylinder axis considered in the present paper. As far as the propagation is concerned, Roquejoffre proved in [48, 49] that, for this class of problems, the solution of the initial value problem converges at long times to a particular traveling wave solution under a rather general set of assumptions (see also [40]).

In the case of the Dirichlet boundary conditions, the first results on existence go back to Gardner who proved existence of front-like solutions for (1.1) for bistable nonlinearity in the absence of a flow on a strip in $\mathbb{R}^2$ [28]. A more general study of existence of traveling waves connecting two stable equilibria in cylinders was carried out by Vega, still in the absence of a flow [59]. More recently, Heinze presented a treatment of mixed types (including nonlinear) of boundary conditions, again, for bistable nonlinearities and in the absence of a flow, using a variational formulation [32]. Also, Freidlin used probabilistic methods to study front propagation in cylinders with Neumann and Dirichlet boundary conditions for KPP-type nonlinearities and in the presence of a shear flow [27].

More recently, in a joint work with Lucia we used a variational formulation to establish existence of certain front-like solutions in the general class of gradient reaction-diffusion systems and no advection [37, 38]. Our analysis covered previous results for scalar problems with bistable and monostable nonlinearities, as well as giving new existence results for more general classes of nonlinearities, as well as for systems of reaction-diffusion equations. Here we extend our method of [38] to problems with advection and concentrate on the scalar equation, for which much stronger results can be obtained by combining our variational approach with the applications of the Maximum Principle. As a result, we obtain a rather complete characterization of front-like traveling wave solutions of (1.1), (1.4), providing a unified treatment for different kinds of nonlinearity in (1.1).

Our approach to the problem of propagation relies on the observation, first made in the case of gradient reaction-diffusion systems [44] and generalized here to the considered class of reaction-diffusion-advection problems, that (1.1) written in the reference frame moving with speed c along the axis of the cylinder is a gradient flow in the exponentially weighted L^2-space:

$$u_t = -e^{-cz-\varphi(y)} \frac{\delta \Phi_c[u]}{\delta u}, \tag{1.5}$$

generated by the functional in (3.3). In particular, traveling wave solutions with the right decay at $z = +\infty$ are critical points of Φ_c, and so one can try to use direct methods of Calculus of Variations to look for the minimizers of this functional. Such an analysis was performed by the authors in the joint work with Lucia [37, 38] for the case in which advection is absent. Here we extend these results on existence of traveling waves to the present case of scalar reaction-diffusion equations with transverse advection by a potential flow. In fact, due to the scalar nature of the problem, a number of stronger statements than in [38] can be made about the obtained traveling wave solutions. In particular, we prove that the obtained solutions are unique (up to translations) and are monotone decreasing in z. Hence, the solution connects $u = 0$ at $z = +\infty$ to a critical point of the functional E from (3.10) that has negative energy at $z = -\infty$, and, furthermore, the asymptotic exponential decay at both ends can be characterized precisely.

In the absence of non-trivial minimizers for the functional in (3.3), we were able to prove existence of a traveling wave solution with a "minimal speed", a result that generalizes the known results for the KPP-type nonlinearities. In this case, too, we are able to establish monotonicity in z and asymptotic decay at both ends of the cylinder. In general, there is no uniqueness, but under certain extra assumptions on the equilibria with negative energy for (3.10) uniqueness (up to translations) can be established as well.

Where our variational method becomes especially powerful is the sharp reaction zone limit of (2.16) arising in combustion problems [16,25,64]. There we can pass to the sharp reaction zone limit directly in (3.3) to obtain a free boundary problem that has been investigated earlier for this kind of problems [2,7,55]. Using our variational approach, we obtain a novel variational formulation, a kind of an area functional, that gives an upper bound for the propagation speed of the traveling waves. Importantly, together with a suitable choice of a trial function, this novel variational formulation also provides a matching lower bound for the propagation speed in the limit of vanishing front curvature, thus giving a rigorous justification to the Markstein model of a flame front [41]. This formulation for the case of an attached flame is also related to the functional introduced by Joulin [33]. We also prove convergence of the regularizing approximations to the solutions of the free boundary problem in the sharp reaction zone limit.

We conclude our study by showing that in all cases the speed of the traveling wave obtained by our variational approach is in fact the asymptotic propagation speed of the leading edge of the solutions of the initial value problem for (1.1) with initial data decaying sufficiently rapidly to zero at $z = +\infty$ and remaining sufficiently far from zero at $z = -\infty$. This demonstrates that the obtained variational traveling waves do in fact play a crucial role in the propagation phenomena governed by the considered type of the reaction-diffusion-advection problems. We note that our analysis makes only a limited use of the Maximum Principle, and, instead, relies mostly on the gradient structure of (1.1) given by (1.5). Thus, our analysis is quite different from that of most of the studies of these problems and provides an analytical alternative to the earlier Maximum Principle-based studies. Moreover, our method allows to obtain good estimates of the propagation speeds (or propagation failure) without constructing sophisticated upper and lower solutions. We demonstrate this point explicitly with a few numerical examples.

This paper is organized as follows. In Sec. 2, we introduce a modeling setup which leads to the problem we are analyzing. We also discuss the singular limit arising in combustion problems which is the subject of Secs. 5 and 6 later on. In Sec. 3, we present all the basic assumptions used throughout the paper. In Sec. 4, we present our main results, given by Theorems 4.3, 4.9, and 4.11 on the existence and properties of certain special traveling wave solutions, which play the key role for the propagation results of Sec. 7. Next, in Sec. 5 we prove existence of traveling wave solutions for the free boundary problem arising in the sharp reaction zone limit, Theorem 5.1, and establish a number of results about the upper and lower bounds for the propagation speed. Then, in Sec. 6 we prove convergence of the traveling wave solutions for the regularizing approximations to the solution of the free boundary problem in the sharp reaction zone limit. In the next section, Sec. 7, we establish general propagation results for the leading edge of the solutions of the initial value problem in (1.1), see Theorem 7.8. In Sec. 8, we illustrate our findings with a few numerical examples. Finally, in

Sec. 9 we summarize the obtained results and compare them with other studies in the literature, and then discuss some open problems.

Notation. Throughout the paper C^k, C_0^∞, $C^{k,\alpha}$ denote the usual spaces of continuous functions with k continuous derivatives, smooth functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0,1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. Unless it is otherwise clear from the context, "$\cdot$" denotes a scalar product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. The symbol ∇ is reserved for the gradient in $\mathbb{R}^n$, while ∇_y stands for the gradient in $\Omega \subset \mathbb{R}^{n-1}$. Similarly, the symbol Δ stands for the Laplacian in $\mathbb{R}^n$, and Δ_y for the Laplacian in Ω. By a classical solution of (4.1) we mean a function $u \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$ that satisfies this equation with a given value of $c > 0$ and the boundary conditions in (1.4). The classical solution of (1.1) is understood to be a $C_1^2(\Sigma \times (0,\infty)) \cap C^0(\overline{\Sigma} \times [0,\infty))$ function [22]. The numbers C, K, M, λ, etc., will denote generic positive constants.

2 Model

In this section we give the physical motivation for a particular modeling setup that leads to the problem analyzed in this paper. We note that, although the derivation is done in the context of a thermodiffusional model of combustion [16, 64], similar modeling is applicable in a wider context, in particular, in the case of chemical reactions in gel reactors (see, e.g., [13]).

Let us recall briefly the thermodiffusional model of laminar flames. Let $n = n(x,t)$ be the fuel concentration and $T = T(x,t)$ the temperature of the gas mixture. The governing equations of this model take the following form:

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = D\Delta n - \bar{\nu}_0 n e^{-E_a/T}, \tag{2.1}$$

$$c\rho \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \kappa \Delta T + \bar{\nu}_0 n E e^{-E_a/T}. \tag{2.2}$$

Here, $\mathbf{v}$ is the velocity field of the imposed advective flow; D is the fuel diffusion coefficient; c, ρ, κ are the specific heat, density, and heat conductance of the mixture, respectively; $\bar{\nu}_0$ is the frequency parameter, E is the reaction heat, and E_a is the activation energy (we use energy units to measure temperature).

The portion $\partial\Sigma_+$ corresponds to the fuel inlet ($\mathbf{v} \cdot \nu|_{\partial\Omega_+} < 0$), hence the fuel concentration there will be high and temperature low; the portion $\partial\Sigma_-$ corresponds to the products outlet ($\mathbf{v} \cdot \nu|_{\partial\Omega_-} > 0$), hence the fuel concentration there will be low and temperature high:

$$T(y,z,t)\big|_{\partial\Sigma_\pm} = T_\pm(y), \qquad n(y,z,t)\big|_{\partial\Sigma_\pm} = n_\pm(y). \tag{2.3}$$

On the other hand, the portion $\partial\Sigma_0$ is impermeable ($\nu \cdot \mathbf{v}|_{\partial\Sigma_0} = 0$):

$$\nu \cdot \nabla T(y,z,t)\big|_{\partial\Sigma_0} = 0, \qquad \nu \cdot \nabla n(y,z,t)\big|_{\partial\Sigma_0} = 0. \tag{2.4}$$

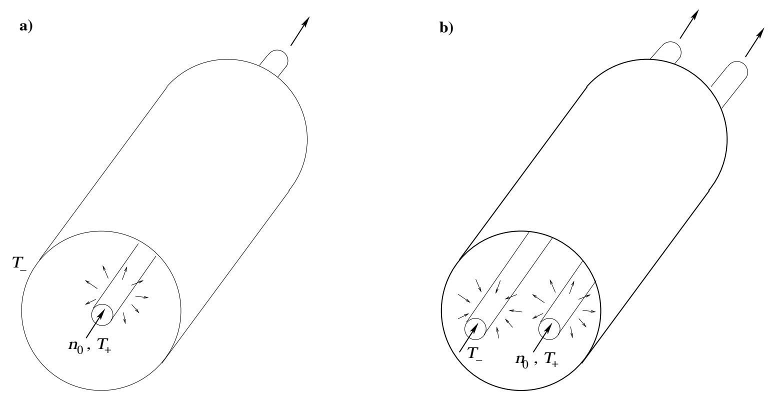

Figure 1: An illustration of a physical setup leading to (1.1).

A simple illustrative example of the system's geometry would be a pair of coaxial perforated pipes, then the reactor Σ is the space between the pipes, see Fig. 1(a). The interior pipe carries cold fuel, with (constant) temperature $T = T_+$ and fuel concentration $n = n_0$, which then enters the reactor, the products (or unburned fuel) escape through the wall of the outer pipe which is in contact with inert gas mixture on the outside, with $T = T_-$ and $n = 0$. Or consider an isolated pipe with two smaller pipes inside, one supplying the fuel and the second used as the exhaust, in this case there are no-flux boundary conditions on the outermost pipe, see Fig. 1(b). It is also natural to assume in such a setup that the advective flow is incompressible, in this case φ is harmonic in Ω. Let us also note that the problems of ignition in a slit burner [52] and propagation of an edge flame [19, 36, 56] fall naturally within our framework.

After an appropriate rescaling (2.1) and (2.2) can be written in the following dimensionless form

$$n_t + \mathbf{v} \cdot \nabla n = \mathrm{Le}^{-1} \Delta n - n e^{-a/T}, \tag{2.5}$$

$$T_t + \mathbf{v} \cdot \nabla T = \Delta T + n e^{-a/T}, \tag{2.6}$$

where we introduced dimensionless parameters

$$\mathrm{Le} = \frac{\kappa}{c\rho D}, \qquad a = \frac{c\rho E_a}{E n_0}, \tag{2.7}$$

where n_0 is the characteristic fuel density. As usual, when $\mathrm{Le} = 1$, we can add up these equation to eliminate one of the variables. Denoting $w = T + n$ and assuming a steady state for w, we find that $w = w(y)$ and satisfies

$$\mathbf{v} \cdot \nabla w = \Delta w, \qquad w\big|_{\partial\Sigma_\pm} = T_\pm + n_\pm, \quad \nu \cdot \nabla w\big|_{\partial\Sigma_0} = 0. \tag{2.8}$$

Substituting the solution of this equation back into (2.5) and introducing u such that

$$T(y,z,t) = u(y,z,t) + u_0(y), \tag{2.9}$$

where $u_0 = u_0(y)$ is a solution of

$$\Delta u_0 - \mathbf{v} \cdot \nabla u_0 + (w - u_0) e^{-a/u_0} = 0, \ u_0\big|_{\partial\Sigma_\pm} = T_\pm, \ \nu \cdot \nabla u_0\big|_{\partial\Sigma_0} = 0, \quad (2.10)$$

we obtain (1.1), where f is the (generally, y-dependent) combustion-type nonlinearity

$$f(u, y) = (w(y) - u_0(y) - u) e^{-a/(u_0(y)+u)} - (w(y) - u_0(y)) e^{-a/u_0(y)}. \quad (2.11)$$

An important limiting case that arises in the combustion problems is that of the sharp reaction zone asymptotics [16, 25, 64]. This limit is due to the specific Arrhenius nature of the nonlinearity in combustion and occurs in the case of large activation energy, in which the dimensionless parameter $a \gg 1$. To see this, assume first that the quantities in (2.8) are $T_+ = T_- = 0$ (for simplicity, we do not consider the effect of finite temperature of the cold gas) and $n_+ = n_- = 1$, so both the inlet and the exhaust are maintained at the same fuel concentrations. These boundary conditions immediately imply that $w = 1$ and $u_0 = 0$ in Ω (we define the Arrhenius factor to be zero at $u = 0$). So, $f(u) = (1 - u)e^{-a/u}$ and is independent of y.

To proceed, let us set $\varepsilon = a^{-1}$, so that $\varepsilon \ll 1$, and introduce

$$f_\varepsilon(u) = C\varepsilon^{-2}(1 - u)e^{-\varepsilon^{-1}(1-u)(1+(1-u)/u)}, \quad (2.12)$$

with

$$C = \frac{2\varepsilon^4}{\varepsilon(\varepsilon + 1) - (2\varepsilon + 1)e^{\frac{1}{\varepsilon}}\Gamma\left(0, \frac{1}{\varepsilon}\right)}, \qquad \lim_{\varepsilon \to 0} C(\varepsilon) = 1, \quad (2.13)$$

where $\Gamma(a, z)$ is the incomplete Gamma-function. With these definitions we have

$$\int_0^1 f_\varepsilon(u) du = 1. \quad (2.14)$$

We also note that $f_\varepsilon(u)$ is an extremely rapidly decaying function of u. Therefore, unless u is very close to 1, for realistic values of $\varepsilon \ll 1$ the value of $f_\varepsilon(u)$ may be so small that the physical assumptions used to derive (1.1) are no longer valid. This motivates a common physical approximation used in the combustion models to truncate the function f_ε at some $u = u^* < 1$, called the "ignition temperature", and set $f_\varepsilon(u) \equiv 0$ for all $u \in [0, u^*]$. To be consistent with (2.12) and (2.14), one chooses $1 - u^* = O(\varepsilon)$. We note that the latter assumption turns out to be rather crucial for the analysis of regularity of the solutions in the limit $\varepsilon \to 0$ [7].

Now, let us do a rescaling

$$x \to a\sqrt{C}e^{a/2}\, x, \qquad t \to Ca^2e^a\, t. \quad (2.15)$$

Then, the assumptions above can be reformulated in the following version of (1.1):

$$u_t + \mathbf{v} \cdot \nabla u = \Delta u + f_\varepsilon(u), \qquad f_\varepsilon(u) = \varepsilon^{-1} g\left(\frac{1 - u}{\varepsilon}\right), \quad (2.16)$$

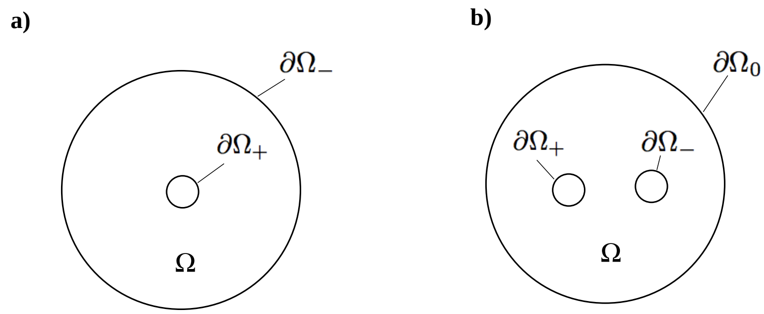

Figure 2: Possible geometry of the domain Ω.

where $g \geq 0$, $\text{supp}(g) = [0,1]$, and $g \in C^1$ on its support, with $\int_0^1 g(u)du = 1$. When $\varepsilon \to 0$, equation (2.16) describes flame propagation in the sharp reaction zone limit.

3 Preliminaries and main hypotheses

In this section, we summarize all the hypotheses used in this paper in the analysis of (1.1). Throughout the paper, $\Omega \subset \mathbb{R}^{n-1}$ is assumed to be a bounded, connected (possibly multiply connected) open set with boundary of class C^2. Of course, the physically relevant case is $n \leq 3$, but we will not make any restrictions on the dimension n, as long as it is not essential for the arguments. We also assume that $\partial\Omega_\pm$ and $\partial\Omega_0$ are a collection of finitely many (possibly one) closed disjoint portions of $\partial\Omega$ (see Fig. 2 for the sketch corresponding to Fig. 1).

Now we discuss the assumptions on the nonlinearity $f(u,y)$. Our method is quite general, and so we do not need to explicitly prescribe the type of the nonlinearity in our problem. We basically need to assume that $f(\cdot,y)$ is sufficiently regular on some compact subset of $\mathbb{R}$ which is an invariant set with respect to the evolution governed by (1.1). Since the Maximum Principle holds for (1.1), without the loss of generality we may assume that $u(x,t) \in [0,1]$, as long as:

(H1) The function $f : [0,1] \times \overline{\Omega} \to \mathbb{R}$ satisfies

$$f(0,y) = 0, \qquad f(1,y) \leq 0. \qquad \forall y \in \Omega. \tag{3.1}$$

(H2) For some $\gamma \in (0,1)$

$$f \in C^{0,\gamma}([0,1] \times \overline{\Omega}), \qquad f_u \in C^{0,\gamma}([0,1] \times \overline{\Omega}), \qquad \varphi \in C^{1,\gamma}(\overline{\Omega}), \tag{3.2}$$

where $f_u = \partial f / \partial u$.

Even though we are not making any particular assumptions about the nonlinearity, let us briefly discuss the types of function f that are usually considered

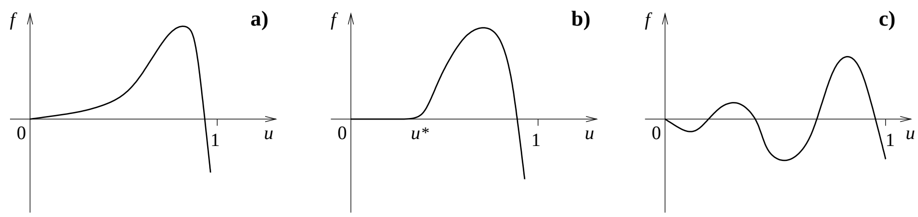

Figure 3: Three examples of the types of the nonlinearity f arising in applications.

in the present context (for the sake of simplicity, we suppress the y-dependence of f here). Specifically, in combustion problems it is meaningful to assume that $f(u) \geq 0$ for all $u \in [0, u_{\max}]$, and that, furthermore, f_u is an increasing function near the origin, Fig. 3(a). Let us point out that in the combustion context f_u can be very small at $u = 0$, which motivates the "ignition temperature" nonlinearity type, Fig. 3(b) (see also the discussion at the end of Sec. 2). On the other hand, our method allows to treat pretty arbitrary nonlinearities, e.g. the ones with multiple equilibria, as in Fig. 3(c), especially if f is decreasing in the neighborhood of zero. Of course, our nonlinearity is allowed to depend on y and can furthermore change the type from one point to the other in Ω.

The starting point of our variational approach is the functional

$$\Phi_c[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2} |\nabla u|^2 + V(u, y) \right) dx, \tag{3.3}$$

where

$$V(u, y) = \begin{cases} 0, & u < 0, \\ -\int_0^u f(s, y)\, ds, & 0 \leq u \leq 1, \\ -\int_0^1 f(s, y)\, ds, & u > 1. \end{cases} \tag{3.4}$$

From the definition of V and the assumptions on f it readily follows that $|V(u)| \leq Cu^2$, and so $\Phi_c[u]$ is naturally defined in an exponentially weighted Sobolev space $H^1_c(\Sigma)$ (for shortness we do not explicitly mention φ, which is part of the definition of $H^1_c(\Sigma)$). Formally, let $\mathcal{D}(\Sigma)$ be the subspace of $C^\infty(\Sigma)$ defined by the restrictions to Σ of all the functions $u \in C^\infty_0(\mathbb{R}^n)$ which vanish on $\partial\Sigma_\pm$. Then

Definition 3.1. *For $c > 0$, denote by $H^1_c(\Sigma)$ the completion of $\mathcal{D}(\Sigma)$ with respect to the norm*

$$||u||^2_{H^1_c(\Sigma)} = ||u||^2_{L^2_c(\Sigma)} + ||\nabla u||^2_{L^2_c(\Sigma)}, \qquad ||u||^2_{L^2_c(\Sigma)} = \int_\Sigma e^{cz+\varphi(y)} |u|^2 dx. \tag{3.5}$$

These are the spaces in which we will consider both the minimizers of Φ_c and the solutions of the initial value problem for (1.1).

Let us mention an important general property of the spaces $H^1_c(\Sigma)$ which is an analogue of the Poincaré inequality and will be needed to establish the existence result (for the proof we refer to [38]).

Lemma 3.2. *For all $u \in H^1_c(\Sigma)$, we have*

$$\frac{c^2}{4}\int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u^2 dydz \leq \int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u_z^2 dydz, \tag{3.6}$$

$$\int_\Omega e^{\varphi(y)}u^2(y,R)dy \leq \frac{e^{-cR}}{c}\int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u_z^2\, dydz, \tag{3.7}$$

for any $R \in \mathbb{R}\cup\{-\infty\}$.

We also have the following obvious inclusions for spaces $H^1_c(\Sigma)$ with different values of c:

Lemma 3.3. *Let $c' > c > 0$ then*

$$H^1_{c'}(\Sigma) \cap W^{1,\infty}(\Sigma) \subset H^1_c(\Sigma) \cap W^{1,\infty}(\Sigma). \tag{3.8}$$

We now turn to the hypothesis that is crucial to the existence of the minimizers of Φ_c:

(H3) There exist $c > 0$, satisfying $c^2 + 4\nu_0 > 0$, where

$$\nu_0 = \min_{\substack{\psi\in H^1(\Omega)\\ \psi|_{\partial\Omega_\pm}=0}} R(\psi), \tag{3.9}$$

with $R(\psi)$ given by (4.5), and $u \in H^1_c(\Sigma)$, such that $\Phi_c[u] \leq 0$ and $u \not\equiv 0$.

This type of condition was already used in [38] in the context of Ginzburg-Landau problems as a sufficient condition for existence of variation traveling waves, and is needed for proving sequential lower semicontinuity of Φ_c in the weak topology of $H^1_c(\Sigma)$. What we will show here, however, is that for scalar equations this condition is also necessary for existence of minimizers of Φ_c.

Let us also introduce an auxiliary functional

$$E[v] = \int_\Omega e^{\varphi(y)}\left(\frac{1}{2}|\nabla_y v|^2 + V(v,y)\right)dy, \tag{3.10}$$

defined for all $v \in H^1(\Omega)$ satisfying Dirichlet boundary conditions on $\partial\Omega_\pm$. By regularity of V and φ the critical points of E satisfy

$$\Delta_y v + \nabla_y\varphi\cdot\nabla_y v + f(v,y) = 0, \qquad v|_{\partial\Omega_\pm} = 0, \quad \nu\cdot\nabla v|_{\partial\Omega_0} = 0. \tag{3.11}$$

Clearly, $v = 0$ is a critical point of E, and in general there may exist a non-trivial minimizer of E over all $v \in H^1(\Omega)$ subject to $v|_{\partial\Omega_\pm} = 0$, call it v_0 [20]. In this case necessarily $v_0 > 0$ and $E[v_0] \leq 0$. Thus, existence of a non-trivial minimizer of E generically guarantees existence of a critical point of E with negative energy.

Note that in general not much is known about the properties of the critical points of E, in particular, about their stability or uniqueness. This information would be needed for a more precise analysis of the multiplicity of the traveling wave solutions of (1.1). In a number of simple cases (e.g. radial symmetry, or one-dimensional domain Ω) these properties can be established from the analysis of the Euler-Lagrange equation for E (see, e.g. [29]). On the other hand, because Ω is bounded, one would expect non-degeneracy (in the sense of second variation) to be a generic property for critical points of E. Furthermore, in the combustion context it would be meaningful to suppose that there is only one positive local minimizer with negative energy. This would basically imply that the shape of a two-dimensional flame in Ω is uniquely determined by the flow generated by φ and the shape of the boundary.

4 Existence and properties of traveling waves

In this section we analyze existence of traveling wave solutions for (1.1). A traveling wave solution is a pair $(c, \bar{u})$, with $c > 0$, such that $u(x,t) = \bar{u}(y, z-ct)$ solves (1.1). Substituting this form into (1.1), we obtain an elliptic equation for $\bar{u}$ with the respective boundary conditions

$$\Delta\bar{u} + c\bar{u}_z + \nabla_y\varphi \cdot \nabla_y\bar{u} + f(\bar{u}, y) = 0, \qquad \bar{u}\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla\bar{u}\big|_{\partial\Sigma_0} = 0. \quad (4.1)$$

Note that (4.1) may in general have many solutions [12, 23, 43]. In the context of the initial value problem for (1.1) one is interested in the particular type of traveling waves in the form of fronts that invade the $u = 0$ equilibrium at $z = +\infty$. From the basic energy estimates for (4.1), one expects the solution to connect two distinct equilibria: $v_+ = 0$ at $z = +\infty$ and $v_- = v(y)$ at $z = -\infty$, where v is a solution of (3.11), when $c > 0$ [12, 23, 59]. The speed c of such a front is part of the problem of finding solutions of (4.1).

Suppose there exists a solution of (4.1) with a particular speed $c > 0$. Linearlizing (4.1) with respect to $u = 0$, we obtain that (here we present a formal discussion of the decay of the solutions in order to clarify the key issues, these statements will be justified later on)

$$\bar{u}(y,z) \sim \sum_k a_k \psi_k(y) e^{-\lambda_k z}, \quad (4.2)$$

which describes the asymptotic behavior of the traveling wave solution at $z = +\infty$, provided that all $\lambda_k > 0$. Here λ_k satisfy a quadratic equation

$$\lambda_k^2 - c\lambda_k - \nu_k = 0. \quad (4.3)$$

where ν_k are the eigenvalues of

$$\Delta_y\psi_k + \nabla_y\varphi \cdot \nabla_y\psi_k + f_u(0,y)\psi_k + \nu_k\psi_k = 0, \quad (4.4)$$

with the same boundary conditions as in (4.1). The eigenvalue problem in (4.4) can be easily characterized.

Proposition 4.1. *There exist a countable set of eigenvalues $\{\nu_k\}$ and a complete set of orthonormal (in $L^2(\Omega; e^{\varphi(y)}dy)$) eigenfunctions ψ_k for problem (4.4). All ν_k are real, and $\nu_0 < \nu_1 \le \nu_2 \le \dots \nu_k \to \infty$. One can choose $\psi_0 > 0$ in Ω, conversely all the other eigenfunctions change sign for $k \ge 1$.*

Proof. The existence of an increasing sequence of real eigenvalues converging to $+\infty$ follows from the spectral representation theorem for compact self-adjoint operators (see for instance [15, Theorem VI.11]). The fact that ν_0 has multiplicity one follows from the characterization of ψ_0 as a minimizer of the Rayleigh quotient

$$R(\psi) = \frac{\int_\Omega e^{\varphi(y)}(|\nabla_y \psi|^2 - f_u(0,y)\psi^2)dy}{\int_\Omega e^{\varphi(y)}\psi^2 dy}, \tag{4.5}$$

which also gives $\psi_0 > 0$, by Strong Maximum Principle. Since the other eigenvectors are orthogonal to ψ_0, they must necessarily change sign. □

Remark 4.2. *Notice that if $\nu_0 \neq 0$, then $v = 0$ is an isolated critical point for the functional E in the cone $C = \{v \in H^1(\Omega) : \ v \ge 0\}$.*

Proof. Assume by contradiction that there exists a sequence of critical points $v_n \to 0$ in $H^1(\Omega)$, such that $v_n \ge 0$. Letting $\tilde{v}_n = v_n/\|v_n\|_{H^1(\Omega)}$, since each v_n solves (3.11), by elliptic regularity we have the estimate

$$\|\tilde{v}_n\|_{H^1(\Omega)} = 1 \qquad \|\tilde{v}_n\|_{H^2(\Omega)} \le C, \tag{4.6}$$

for some $C > 0$. In particular, there exists a function $\tilde{v} \in H^1(\Omega)$, with $\|\tilde{v}\|_{H^1} = 1$ and $\tilde{v} \ge 0$, such that $\tilde{v}_n \to \tilde{v}$ in $H^1(\Omega)$. Recalling that $\tilde{v}_n$ satisfies the equation

$$\Delta_y \tilde{v}_n + \nabla_y \varphi \cdot \nabla \tilde{v}_n + \frac{1}{\|v_n\|_{H^1(\Omega)}} f\left(\|v_n\|_{H^1(\Omega)}\tilde{v}_n, y\right) = 0, \tag{4.7}$$

passing to the limit as $n \to +\infty$, we obtain that $\tilde{v}$ solves (4.4) with $\nu_k = 0$, thus contradicting Proposition 4.1. □

In the following, we always assume that $\psi_0 > 0$. Then, in order for $\bar{u}$ to remain positive for all $z > 0$ we need $a_0 > 0$ and $\lambda_0 < \lambda_k$ for all $k > 0$. Let us first consider the simpler case of $\nu_0 > 0$, which corresponds to the situation in which $u = 0$ is locally stable with respect to (1.1). In this case (4.3) has a unique positive solution for each k, and, furthermore, λ_k are increasing with k. Therefore, the asymptotic behavior of $\bar{u}$ is given by $a_0\psi_0(y)e^{-\lambda_0^+ z}$, where $a_0 > 0$ and $\lambda_k^\pm = \lambda_\pm(c, \nu_k)$ with

$$\lambda_\pm(c, \nu_k) = \frac{c \pm \sqrt{c^2 + 4\nu_k}}{2}. \tag{4.8}$$

Note that $\lambda_0^+ > \frac{c}{2}$, and so these solutions are expected to lie in the exponentially weighted Sobolev space $H_c^1(\Sigma)$.

On the other hand, the case of $\nu_0 < 0$, when $u = 0$ is unstable, requires a more careful consideration. First of all, it is clear that we should have $c^2 + 4\nu_0 \geq 0$ in order for $\bar{u}$ to remain positive (otherwise the approach to zero is oscillatory due to the imaginary part of λ_k). However, when $c^2 + 4\nu_0 > 0$, there are two positive solutions of (4.3) for λ_0, according to (4.8). In fact, one would generically expect the decay of the solution to be governed by $\lambda_0^- = \lambda_-(c, \nu_0)$, since $\lambda_-(c, \nu_0) < \lambda_+(c, \nu_0)$ in this case. On the other hand, if the solution is also known to lie in $H_c^1(\Sigma)$, then λ_0^- is not allowed, since $\lambda_0^- < \frac{c}{2}$ would make $\bar{u}$ fail to lie in $H_c^1(\Sigma)$. Therefore, those traveling wave solutions that lie in $H_c^1(\Sigma)$ are expected to have a non-generic exponential decay $a_0 \psi_0(y) e^{-\lambda_0^+ z}$, with $a_0 > 0$. This is still true in the case $\nu_0 = 0$ for exponentially decaying solutions.

One can repeat the above arguments to study the behavior of the solution at $z = -\infty$. Linearizing around $u = v(y)$, we obtain $u - v \sim \sum_k \tilde{a}_k \tilde{\psi}_k(y) e^{-\tilde{\lambda}_k z}$ and

$$\Delta_y \tilde{\psi}_k + \nabla_y \varphi \cdot \nabla_y \tilde{\psi}_k + f_u(v, y) \tilde{\psi}_k + \tilde{\nu}_k \tilde{\psi}_k = 0. \tag{4.9}$$

Here we should require that $\tilde{\lambda}_k < 0$, where $\tilde{\lambda}_k$ satisfy (4.3) with $\tilde{\nu}_k$ instead of ν_k. Assuming that all $\tilde{\nu}_k \neq 0$, one sees immediately that $\tilde{a}_k = 0$ for all $\tilde{\nu}_k < 0$. If, furthermore, it is known that $\bar{u} - v < 0$ for large negative z, then we must have $\tilde{\nu}_0 > 0$ and $\tilde{a}_0 < 0$, and choose $\tilde{\lambda}_0 = \tilde{\lambda}_0^- = \lambda_-(c, \tilde{\nu}_0)$. In other words, under the assumptions of non-degeneracy of v and approach from below, the equilibrium v is necessarily a local minimum of E.

If $\bar{u} \in H_c^1(\Sigma)$, then, at least formally, $\bar{u}$ is a critical point of the functional Φ_c, since the first variation of Φ_c is

$$\begin{aligned} \delta\Phi_c[u] &= \int_\Sigma e^{cz + \varphi(y)} \left(\nabla u \cdot \nabla \delta u + V_u(u, y) \delta u \right) dx \\ &= -\int_\Sigma e^{cz + \varphi(y)} \Big(\Delta u + c u_z + \nabla_y \varphi \cdot \nabla_y u + f(u, y) \Big) \delta u \, dx, \end{aligned} \tag{4.10}$$

where we integrated by parts, using the boundary conditions from (1.1), and assumed that $0 \leq u(x) \leq 1$. We call this type of traveling wave solutions *variational traveling waves* [38, 44]. Among these solutions, of special interest are the traveling wave solutions which are in fact *minimizers* of Φ_c in $H_c^1(\Sigma)$. Let us note that existence of a minimizer $\bar{u}$ of Φ_c implies that

$$\Phi_c[\bar{u}] = 0. \tag{4.11}$$

This follows immediately from the way the functional Φ_c transforms under translations

$$\Phi_c[u(y, z - a)] = e^{ca} \Phi_c[u(y, z)], \tag{4.12}$$

and the fact that $\Phi_c[\bar{u}]$ should not change under infinitesimal translations of $\bar{u}$. We also point out that for the same reason (4.11) should in fact hold for any critical point of Φ_c and, hence, for any variational traveling wave.

Let us note that not all variational traveling waves can be minimizers of Φ_c, and not all traveling wave solutions, of course, have to be variational. Nevertheless, as was shown in [37, 44], for a large class of nonlinearities and sufficiently localized initial data only the variational traveling waves can be selected as the long-time attractors for the initial value problem governed by (1.1). It may also happen that the minimizer of Φ_c is the only variational traveling wave among all traveling wave solutions satisfying $0 < \bar{u}(x) < 1$ in Σ, hence, the only candidate for the long-time asymptotic behavior of the solutions of the initial value problem.

Since variational traveling waves play a key role in the propagation phenomena governed by (1.1), we will concentrate our efforts on establishing existence and uniqueness of these solutions. Later, in Sec. 7, we will show that their speed in fact determines the asymptotic long time propagation speed for the solutions of the initial value problem for (1.1) with sufficiently localized initial data (for more precise definitions and results, see Sec. 7). Below is our main result concerning the existence and properties of variational traveling waves for (1.1).

Theorem 4.3. *Under hypotheses (H1)–(H3), there exists a unique* $c^\dagger \in \mathbb{R}$ *such that* $c^\dagger \geq c > 0$ *(where* c *is the "trial velocity" given by assumption (H3)), and* $\bar{u} \in H^1_{c^\dagger}(\Sigma)$, $\bar{u} \not\equiv 0$, *such that*

(i) $\bar{u} \in C^2(\Sigma) \cap W^{1,\infty}(\overline{\Sigma})$, $\bar{u}$ *solves (4.1) with* $c = c^\dagger$, *and* $\bar{u}$ *is a minimizer of* $\Phi_{c^\dagger}$.

(ii) $\bar{u}(y, z)$ *is strictly monotone decreasing in* z *for all* $y \in \Omega$, $\lim_{z\to+\infty} \bar{u}(\cdot, z) = 0$ *in* $C^1(\overline{\Omega})$, *and* $\lim_{z\to-\infty} \bar{u}(\cdot, z) = v$ *in* $C^1(\overline{\Omega})$, *where* v *is a critical point of* E, *with* $E[v] < 0$ *and* $0 < v \leq 1$ *in* Ω.

(iii) $\bar{u}(y, z) = a_0\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)z} + O(e^{-\lambda z})$, *with some* $a_0 > 0$ *and* $\lambda > \lambda_+(c^\dagger, \nu_0)$, *uniformly in* $C^1(\overline{\Omega} \times [R, +\infty))$, *as* $R \to +\infty$.

(iv) $\tilde{\nu}_0 \geq 0$, *moreover, if* $\tilde{\nu}_0 > 0$, *then* $\bar{u}(y, z) = v(y) + \tilde{a}_0\tilde{\psi}_0(y)e^{-\lambda_-(c^\dagger,\tilde{\nu}_0)z} + O(e^{-\lambda z})$, *with some* $\tilde{a}_0 < 0$ *and* $\lambda < \lambda_-(c^\dagger, \tilde{\nu}_0)$, *uniformly in in* $C^1(\overline{\Omega} \times (-\infty, R])$, *as* $R \to -\infty$.

(v) The obtained minimizer $\bar{u}$ *of* $\Phi_{c^\dagger}$ *is unique, up to translations.*

Proof of Part (i)

The existence of a speed $c^\dagger \geq c$, a function $\bar{u} \in H^1_{c^\dagger}(\Sigma)$ minimizing $\Phi_{c^\dagger}$, and the regularity of $\bar{u}$ can be proved exactly as in [38, Theorem 1.1]. We will outline the proof of this statement here, modifying it in a few parts (so as to not to rely on regularity of $\bar{u}$), in order to be able to apply it in Theorem 5.1 of the next section. The idea is to consider constrained minimizers of Φ_c, i.e., find $u_c \in \mathcal{B}_c$, where

$$\mathcal{B}_c = \left\{ u \in H^1_c(\Sigma) : \int_\Sigma e^{cz+\varphi(y)} u_z^2 \, dx = 2 \right\}, \tag{4.13}$$

which satisfies $\Phi_c[u_c] = \inf_{u\in\mathcal{B}_c} \Phi_c[u]$. Note that by definition $u_c \neq 0$. Then, by hypothesis (H3), we would necessarily have $\Phi_c[u_c] \le 0$. Note that $\bar{u}(x) \in [0,1]$ for all $x \in \Sigma$, since for any $u \in H^1_c(\Sigma)$ we have $\Phi_c[\tilde{u}] \le \Phi_c[u]$, where $\tilde{u}$ is the truncation of u:

$$\tilde{u}(x) = \begin{cases} 0, & u(x) < 0, \\ u(x), & 0 \le u(x) \le 1, \\ 1, & u(x) > 1, \end{cases} \tag{4.14}$$

and, in fact, this inequality is strict, unless $\tilde{u} = u$ a.e.

Now, suppose the constrained minimizer u_c exists, and let

$$c^\dagger = c\sqrt{1 - \Phi_c[u_c]}. \tag{4.15}$$

Note that since $\Phi_c[u_c] \le 0$ we have $c^\dagger \ge c$. For any $u \in H^1_{c^\dagger}(\Sigma), u \neq 0$, define $u_a(y,z) = u\left(y, \frac{c(z-a)}{c^\dagger}\right)$ for all $(y,z) \in \Sigma$. Then clearly $u_a \in H^1_c(\Sigma)$, and it is always possible to choose $a \in \mathbb{R}$ such that $u_a \in \mathcal{B}_c$. Assuming that a is chosen this way, we have

$$\begin{aligned} e^{c^\dagger a}\Phi_{c^\dagger}[u] &= \int_\Sigma e^{c^\dagger z + \varphi(y)} \left(\frac{1}{2}|\nabla u(y, z-a)|^2 + V(u(y,z-a), y)\right) dx \\ &= \left(\frac{c}{c^\dagger}\right) \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{1}{2}\left(\frac{c^\dagger}{c}\right)^2 \left(\frac{\partial u_a}{\partial z}\right)^2 + \frac{1}{2}|\nabla_y u_a|^2 + V(u_a, y) \right\} dx \\ &= \frac{{c^\dagger}^2 - c^2}{2c^\dagger c} \int_\Sigma e^{cz+\varphi(y)} \left(\frac{\partial u_a}{\partial z}\right)^2 dx + \left(\frac{c}{c^\dagger}\right)\Phi_c[u_a] \\ &= \left(\frac{c}{c^\dagger}\right)\left(\Phi_c[u_a] - \Phi_c[u_c]\right), \end{aligned} \tag{4.16}$$

where in the computation of the last line in (4.16) we used (4.13) and (4.15). Now, since $\Phi_c[u_c] \le \Phi_c[u]$ for any $u \in \mathcal{B}_c$, we have $\Phi_{c^\dagger}[u] \ge 0$ for all $u \in H^1_{c^\dagger}(\Sigma)$ and, furthermore, the minimum is attained on $\bar{u}(y,z) = u_c\left(y, \frac{c^\dagger z}{c}\right)$. In other words, $\bar{u}$ is a non-trivial minimizer of $\Phi_{c^\dagger}$.

To prove existence of a constrained minimizer u_c, one picks a minimizing sequence on $\mathcal{B}_c$. Since $\varphi \in L^\infty(\Omega)$, all the arguments in the proofs of Propositions 5.5 and 5.6 of [38] remain valid. The only difference is that in Lemma 5.4 of [38] one needs to estimate $\Phi_c[u, (R, +\infty)]$ with the help of (4.5). The fact that $\bar{u}$ is a classical solution of (4.1), together with gradient estimates, follows by standard regularity theory [30] (see [38, Prop. 3.3]). Indeed, as a minimizer of $\Phi_{c^\dagger}$ the function $\bar{u}$ solves

$$\int_\Sigma e^{c^\dagger z + \varphi(y)} \left(\nabla\bar{u}\cdot\nabla\phi - f(\bar{u}, y)\phi\right) dx = 0, \tag{4.17}$$

where $\phi \in H^1_c(\Sigma)$ is an arbitrary test function. This is the weak form of (4.1).

Finally, to prove uniqueness of $c^\dagger$, suppose there exist $c_1^\dagger > c_2^\dagger > 0$, and the corresponding non-trivial minimizers are $\bar{u}_{1,2}$, with $\bar{u}_1 \in H^1_{c_2^\dagger}(\Sigma)$ by Lemma 3.3. Let $\tilde{u}(y,z) = \bar{u}_1\left(y, \frac{c_2^\dagger z}{c_1^\dagger}\right) \in H^1_{c_2^\dagger}$, then

$$\begin{aligned}\Phi_{c_2^\dagger}[\tilde{u}] &= \left(\frac{c_1^\dagger}{c_2^\dagger}\right)\int_\Sigma e^{c_1^\dagger z+\varphi(y)}\left\{\frac{1}{2}\left(\frac{c_2^\dagger}{c_1^\dagger}\right)^2\left(\frac{\partial \bar{u}_1}{\partial z}\right)^2+\frac{1}{2}|\nabla_y \bar{u}_1|^2+V(\bar{u}_1,y)\right\}dx \\ &= \left(\frac{c_1^\dagger}{c_2^\dagger}\right)\left(\Phi_{c_1^\dagger}[\bar{u}_1]-\frac{{c_1^\dagger}^2-{c_2^\dagger}^2}{2{c_1^\dagger}^2}\int_\Sigma e^{c_1^\dagger z+\varphi(y)}\left(\frac{\partial \bar{u}_1}{\partial z}\right)^2 dx\right)<0. \qquad (4.18)\end{aligned}$$

But this contradicts existence of a minimizer for $\Phi_{c_2^\dagger}$, which implies that $\Phi_{c_2^\dagger}[u] \geq \Phi_{c_2^\dagger}[\bar{u}_2] = 0$ for all $u \in H^1_{c_2^\dagger}(\Sigma)$.

Proof of Part (ii)

Let us first prove monotonicity of $\bar{u}$. The idea of the proof is related to the one used to prove uniqueness later on in Part (v). We note that an alternative way of proving monotonicity of the minimizers is via a one-dimensional monotone rearrangement (see, e.g., [46]).

First of all, by repeating the arguments of [38, Proposition 3.3(iii)] we may conclude that $\bar{u}(z,\cdot) \to 0$ in $C^0(\overline{\Omega})$ as $z \to +\infty$ (in fact, $\bar{u}(y,z) \leq Ce^{-\lambda z}$ for some $C > 0$ and $\lambda > 0$). Standard regularity estimates [30] then imply that the convergence of $\bar{u}(y, z+R)$ is in fact in $W^{2,p}(\Omega \times (0,1))$, for all $p > 1$, as $R \to +\infty$. Hence, in particular, $\bar{u}(z,\cdot) \to 0$ in $C^1(\overline{\Omega})$.

Now, for any $a > 0$, let us introduce

$$\bar{u}_1(y,z) = \min(\bar{u}(y,z), \bar{u}(y,z-a)), \qquad (4.19)$$
$$\bar{u}_2(y,z) = \max(\bar{u}(y,z), \bar{u}(y,z-a)). \qquad (4.20)$$

According to (4.11), we have

$$0 = \Phi_{c^\dagger}[\bar{u}(y,z)] + \Phi_{c^\dagger}[\bar{u}(y,z-a)] = \Phi_{c^\dagger}[\bar{u}_1] + \Phi_{c^\dagger}[\bar{u}_2], \qquad (4.21)$$

and since also $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma)$, it follows that

$$\Phi_{c^\dagger}[\bar{u}_1] = 0, \qquad \Phi_{c^\dagger}[\bar{u}_2] = 0. \qquad (4.22)$$

Hence, $\bar{u}_1$ and $\bar{u}_2$ are also non-trivial minimizers. Now, consider $w = \bar{u}_2 - \bar{u}_1 \geq 0$. In view of hypothesis (H2), w satisfies an elliptic equation

$$\Delta w + c^\dagger w_z + \nabla_y \varphi \cdot \nabla_y w + k(y,z) w = 0, \qquad (4.23)$$

for some $k \in L^\infty(\Sigma)$. Then, according to the argument following (6.8) in [38, Proposition 6.4], which is based on the Strong Maximum Principle, we conclude that either $w = 0$ or $w > 0$ in Σ. The first possibility would imply that $\bar{u}$ is independent of z and, hence, is zero, which is impossible. So, $w > 0$, implying that $\bar{u}(y, z-a) > \bar{u}(y,z)$ for all $x = (y,z) \in \Sigma$. In view of the arbitrariness of $a > 0$, this implies that $\bar{u}$ is strictly monotone decreasing.

Now, as was shown in Part (i), the minimizer $\bar{u}$ takes values from the unit interval. Therefore, by monotonicity of $\bar{u}$, there exists a function $v : \Omega \to \mathbb{R}$, with values $v(y) \in [0,1]$ such that $\bar{u}(y,z) \to v(y)$ for all $y \in \Omega$, hence, again by elliptic regularity, $v \in C^1(\overline{\Omega})$ and $u(\cdot, z)$ converges to v in $C^1(\overline{\Omega})$. For any $R \in \mathbb{R}$, fix a test function $\phi(y,z) = \psi(y)\eta_R(z)$ with arbitrary $\psi \in H^1(\Omega)$, $\psi|_{\partial\Omega_\pm} = 0$ and $\eta_R(z) = \eta_0(z - R) \geq 0$, with $\eta_0 \in C_0^\infty(\mathbb{R})$, then (4.17) reads (here and below the prime denotes differentiation with respect to z)

$$\int_{\mathrm{supp}(\eta)} \int_\Omega e^{c^\dagger z + \varphi(y)} (\psi \bar{u}_z \eta_R' + \eta_R \nabla_y \bar{u} \cdot \nabla_y \psi - f(\bar{u}, y)\eta_R \psi) dy dz = 0. \quad (4.24)$$

Multiplying this equation by $e^{-c^\dagger R}$, passing to the limit $R \to -\infty$ in the integral and using Fubini Theorem, we obtain

$$0 = \int_\Omega e^{\varphi(y)} (\nabla_y v \cdot \nabla_y \psi - f(v,y)\psi) dy, \quad (4.25)$$

which is precisely the Frechet derivative of $E[v]$. Therefore, v is a critical point of E and, furthermore, by standard elliptic regularity, we have $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and v satisfies (3.11).

Let us now show the inequality $E[v] < 0$. First, note that $E[v] = \lim_{z \to -\infty} E[\bar{u}(\cdot, z)]$, and $E[\bar{u}(\cdot, z)]$ is a continuous function of z. Let us show that $E[v] \leq E[\bar{u}(\cdot, z)]$ for all $z \in \mathbb{R}$. Indeed, observe that by (4.11) and Fubini Theorem we have

$$0 = \Phi_{c^\dagger}[\bar{u}] = \int_{-\infty}^{+\infty} e^{c^\dagger z} E[\bar{u}(\cdot, z)] dz + \frac{1}{2} \int_\Sigma e^{c^\dagger z + \varphi(y)} \bar{u}_z^2 dx, \quad (4.26)$$

hence there exists some $z_0 \in \mathbb{R}$ such that $E[\bar{u}(y, z_0)] < 0$. Now, if $E[v] > E[\bar{u}(\cdot, z)]$ for some $z \in \mathbb{R}$, we can choose z_0 to be a minimum of $E[\bar{u}(\cdot, z)]$, in view of the fact that $E[\bar{u}(\cdot, z)] \to 0$ as $z \to +\infty$. Then, taking $\tilde{u}(y,z) = \bar{u}(y, z_0)$ for all $z < z_0$, and $\tilde{u}(y,z) = \bar{u}(y,z)$ for all $z \geq z_0$, for any $y \in \Omega$, we find that $\Phi_{c^\dagger}[\tilde{u}] < 0$, contradicting the fact that $\bar{u}$ is a minimizer. Therefore, $E[v] \leq E[\bar{u}(\cdot, z_0)] < 0$.

Proof of Part (iii)

To obtain the decay of $\bar{u}$ as $z \to +\infty$, we explicitly construct the solution for $z > R$, with R large enough, by expanding it into a Fourier series in terms of the eigenfunctions in (4.4) on the cross sections. The arguments below basically formalize the earlier discussion of the decay of the solution at the beginning of this section (see also [12, 57]).

For any $z \in \mathbb{R}$, introduce

$$a_k(z) = \int_\Omega e^{\varphi(y)} \psi_k(y) \bar{u}(y,z)\, dy. \quad (4.27)$$

By standard $W^{2,p}$ estimates for $\bar{u}$ on slices of Σ [30, 38], we have $a_k \in C^{1,\alpha}(\mathbb{R})$ for any $\alpha \in (0,1)$ and, furthermore, since by Proposition 4.1 the functions ψ_k form a complete orthonormal basis, we obtain [15, Theorem VI.11]

$$\bar{u}(y,z) = \sum_{k=0}^{\infty} a_k(z)\psi_k(y), \qquad \sum_{k=0}^{\infty} a_k^2(z) = \int_\Omega e^{\varphi(y)}\bar{u}^2(y,z)dy, \tag{4.28}$$

where the first series converges in $L^2(\Omega; e^{\varphi(y)}dy)$ for each z. Testing (4.17) with $\phi(y,z) = \psi_k(y)\eta(z)$, where $\eta \in C_0^\infty(\mathbb{R})$ is arbitrary, applying the Fubini Theorem and performing integration by parts, we obtain

$$\int_{-\infty}^{+\infty} e^{c^\dagger z}(a_k'\eta' + (\nu_k a_k + g_k)\eta)dz = 0, \tag{4.29}$$

where we introduced

$$g_k(z) = \int_\Omega e^{\varphi(y)}(f_u(0,y)\bar{u}(y,z) - f(\bar{u}(y,z),y))\psi_k(y)\,dy. \tag{4.30}$$

Note that $g_k \in C^{0,\gamma}(\mathbb{R})$. Again, by standard regularity theory [30], the functions a_k belong to $C^{2,\gamma}(\mathbb{R})$ and satisfy a second-order ordinary differential equation

$$a_k'' + c^\dagger a_k' - \nu_k a_k = g_k. \tag{4.31}$$

Now, think of $g_k(z)$ as the components of a *linear* operator G in the basis of ψ_k's:

$$g_k(z) = \int_\Omega e^{\varphi(y)}(G\bar{u})(y,z)\psi_k(y)dy, \qquad G\bar{u} = (f_u(0,\cdot) - f_u(\tilde{u},\cdot))\bar{u}, \tag{4.32}$$

for some $0 < \tilde{u} < \bar{u}$ by hypothesis (H2), with $\bar{u}$ given by (4.28). Since $f_u(\cdot,y) \in C^{0,\gamma}(\mathbb{R})$, the operator G is a bounded operator from $L^2(\Omega; e^\varphi dy)$ to itself for any fixed $z \in \mathbb{R}$. In the following we freeze $\tilde{u}$ in (4.32) and treat (4.31) as a system of linear ordinary differential equations with $a_k(z) \in l^2$ for all $z \in \mathbb{R}$ or, equivalently, a dynamical system for $\bar{u}(\cdot,z) \in L^2(\Omega; e^{\varphi(y)}dy)$.

Using variation of parameters and keeping in mind that $c^2 + 4\nu_k > 0$ by hypothesis (H3) and Proposition 4.1, one can write the solution for (4.31) in the form

$$\begin{aligned} a_k(z) = a_k^+(R)e^{-\lambda_+(c^\dagger,\nu_k)(z-R)} &+ a_k^-(R)e^{-\lambda_-(c^\dagger,\nu_k)(z-R)} \\ &- \frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}}\int_R^z e^{\lambda_+(c^\dagger,\nu_k)(\xi - z)}g_k(\xi)d\xi \\ &+ \frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}}\int_R^z e^{\lambda_-(c^\dagger,\nu_k)(\xi - z)}g_k(\xi)d\xi, \end{aligned} \tag{4.33}$$

where $a_k^\pm(R)$ are constants of integration satisfying $a_k^+(R) + a_k^-(R) = a_k(R)$. In particular, if $\lambda_-(c^\dagger,\nu_k) < 0$, we have

$$a_k^-(R) = -\frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}}\int_R^{+\infty} e^{\lambda_-(c^\dagger,\nu_k)(\xi - R)}g_k(\xi)d\xi, \tag{4.34}$$

which is obtained by multiplying (4.33) by $e^{\lambda_-(c^\dagger,\nu_k)z}$ and passing to the limit $z \to +\infty$, taking into account boundedness of a_k's and g_k's.

Now, by hypothesis (H2) we have $||G(z)||_{L^2(\Omega;e^{\varphi(y)}dy)} \leq C||\bar{u}(\cdot,z)||^\gamma_{L^\infty(\Omega)}$, hence, in particular, $||G(z)||_{L^2(\Omega;e^{\varphi(y)}dy)} = O(e^{-\mu z})$ for some $\mu > 0$ (see the discussion at the beginning of the proof of Part (ii); this condition is only needed if $\lambda_-(c^\dagger,\nu_k) = 0$ for some k). Then, it is easy to see that with $a_k^+(R)$ fixed for all k and with $a_k^-(R)$ fixed whenever $\lambda_-(c^\dagger,\nu_k) \geq 0$, the mapping defined by (4.33) is a contraction for sufficiently large R in the Banach space with the norm

$$||\bar{u}|| = \sup_{z\in[R,+\infty)} ||\bar{u}(\cdot,z)||_{L^2(\Omega;e^{\varphi(y)}dy)}. \tag{4.35}$$

Indeed, denoting the operator generated by the right-hand side of (4.33) as T and introducing u_1, u_2 as described above, after some straightforward calculations we obtain

$$||T(u_1 - u_2)|| \leq Ce^{-\mu R/2}||u_1 - u_2||. \tag{4.36}$$

In arriving at the last estimate we used the fact that the sequences (ν_k), $(\lambda_+(c^\dagger,\nu_k))$, and $(-\lambda_-(c^\dagger,\nu_k))$ are monotone increasing.

So, T is a contraction, and so for any fixed $a_k^+(R)$, and for any fixed $a_k^-(R)$ corresponding to $\lambda_-(c^\dagger,\nu_k) \geq 0$ there is a unique solution whose $L^2(\Omega;e^{\varphi(y)}dy)$ norm is uniformly bounded on $[R,+\infty)$. Moreover, by the estimate (3.7) of Lemma 3.2, we have $|a_k(z)| \leq Ce^{-c^\dagger z/2}$ for all k, which implies that (4.34) in fact holds whenever $\lambda_-(c^\dagger,\nu_k) \geq 0$ as well, since $\lambda_-(c^\dagger,\nu_k) < \frac{c^\dagger}{2}$ by hypothesis (H3).

Let us now show that the value of $\lambda_+(c^\dagger,\nu_0)$ determines the exponential rate of decay of the solution at $z = +\infty$. For that, it is necessary that (4.34) does not hold with $\lambda_-(c^\dagger,\nu_0)$ replaced with $\lambda_+(c^\dagger,\nu_0)$ and $a_k^-(R)$ replaced by $a_k^+(R)$. Otherwise, there exists $k = k_0$ such that this equation does not hold (the opposite implies that $\bar{u} = 0$ in $\Omega \times (R,+\infty)$). Then $a_k = a_k^+(R)e^{-\lambda(z-R)} + O(e^{-(1+\frac{1}{2}\gamma)\lambda z})$ for all $k_0 \leq k \leq k_1$ for which $\lambda = \lambda_+(c^\dagger,\nu_k)$ (with at least one $a_k^+(R) \neq 0$), and $a_k = O(e^{-(1+\frac{1}{2}\gamma)\lambda z})$ for all other k's. That k_1 is finite follows from the fact that $\lambda_+(c^\dagger,\nu)$ is a strictly monotonically increasing function of ν, and that by Proposition 4.1 the eigenvalues of ν_k have finite multiplicities and $\nu_k \to +\infty$. In view of these estimates we have

$$\bar{u}(y,z) = \sum_{k=k_0}^{k_1} a_k^+(R)e^{-\lambda_+(c^\dagger,\nu_{k_0})(z-R)}\psi_k(y) + o(e^{-\lambda_+(c^\dagger,\nu_{k_0})z}), \tag{4.37}$$

Therefore, by orthogonality of all ψ_k's to $\psi_0 > 0$ for $k \geq k_0$ and the fact that these ψ_k's change sign, we see that $\bar{u}(\cdot,z)$ will become negative somewhere on a set of non-zero measure in Ω. This is clearly impossible, and so we finally obtain the estimate

$$\bar{u}(y,z) = a_0^+(R)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)}\psi_0(y) + O(e^{-\lambda z}), \tag{4.38}$$

with $\lambda = \min\{\lambda_+(c^\dagger,\nu_1),(1+\frac{1}{2}\gamma)\lambda_+(c^\dagger,\nu_0)\} > \lambda_+(c^\dagger,\nu_0)$, in $L^2(\Omega, e^{\varphi(y)}dy)$ for each z. By construction $a_0(z) > 0$, and so $a_k^+(R) > 0$ for large enough R.

Finally, consider the function $w(y,z) = \bar{u}(y,z) - a_0^+(R)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)}\psi_0(y)$ which satisfies a linear equation in $\Omega \times (R,+\infty)$

$$\Delta w + c^\dagger w_z + \nabla_y\varphi \cdot \nabla_y w + f_u(0,y)w - Gw$$
$$= a_0^+(R)G\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)} \tag{4.39}$$

Since for each z both w and the right-hand side of this equation are $O(e^{-\lambda R})$ in $L^2(\Omega\times(R,R+1), e^{\varphi(y)}dy)$ with $\lambda > \lambda_+(c^\dagger,\nu_0)$, standard elliptic regularity theory [30, Theorem 9.13] implies that w is $O(e^{-\lambda R})$ in $W^{2,2}(\Omega\times(R+\frac{1}{4},R+\frac{3}{4}))$ and hence, by Sobolev imbedding, in $L^p(\Omega\times(R+\frac{1}{4},R+\frac{3}{4}), e^{\varphi(y)}dy)$ for some $p > 2$. So, the above estimate in fact holds in $L^p(\Omega\times(R,R+1), e^{\varphi(y)}dy)$. Iterating this argument using $W^{2,p}$ estimates until the space imbeds into $C^1(\overline{\Omega}\times(R+\frac{1}{4},R+\frac{3}{4}))$, we obtain the result.

Proof of Part (iv)

When $\tilde{\nu}_0 > 0$, the proof follows exactly as in Part (iii), where we do not need an a priori estimate on the exponential decay of $\bar{u}(\cdot,z)$ to v as $z \to -\infty$ any more, since all $\tilde{\nu}_k > 0$, and hence $\lambda_-(c^\dagger,\tilde{\nu}_k) < 0$ and $\lambda_+(c^\dagger,\tilde{\nu}_k) > 0$ for all k.

To prove that $\tilde{\nu}_0 < 0$ is impossible, consider the analog of (4.31) with $k = 0$:

$$\tilde{a}_0'' + c^\dagger\tilde{a}_0' - \tilde{\nu}_0\tilde{a}_0 = \tilde{g}_0. \tag{4.40}$$

Observe that since $\bar{u}(\cdot,z) \to v$ uniformly as $z \to -\infty$, by hypothesis (H2) and the fact that $\tilde{\psi}_0 > 0$ and $\bar{u} - v < 0$ we have

$$|\tilde{g}_0(z)| \le C\int_\Omega e^{\varphi(y)}(v(y)-\bar{u}(y,z))^{1+\gamma}\tilde{\psi}_0(y)dy \le \varepsilon|a_0(z)|, \tag{4.41}$$

for any $\varepsilon > 0$, as long as z is sufficiently large negative. It is then easy to see (using e.g. variation of parameters) that (4.40) does not have bounded solutions for $z < R$ when ε is small enough.

Proof of Part (v)

Our proof of uniqueness is based on the argument due to Heinze [32]. Suppose that $\bar{u}_1$ and $\bar{u}_2$ are two non-trivial minimizers of Φ_c. Then, there exists a translation a such that $\bar{u}_1(y^*,z^*) = \bar{u}_2(y^*,z^*-a)$ at some point $x^* = (y^*,z^*) \in \Sigma$. Indeed, if not, then without loss of generality we can assume that $\bar{u}_1(y,z) < \bar{u}_2(y,z-a)$ for all $x = (y,z) \in \Sigma$ and all $a \in \mathbb{R}$. Also, by the result of Part (iii), $\bar{u}_2(y,z-a) \to 0$ as $a \to -\infty$, hence, $\bar{u}_1 = 0$, contradicting the assumption that $\bar{u}_1$ is a non-trivial minimizer. So, $\bar{u}_1(y^*,z^*) = \bar{u}_2(y^*,z^*-a)$, and let us introduce

$$\bar{u}_3(y,z) = \min(\bar{u}_1(y,z), \bar{u}_2(y,z-a)), \tag{4.42}$$
$$\bar{u}_4(y,z) = \max(\bar{u}_1(y,z), \bar{u}_2(y,z-a)). \tag{4.43}$$

Arguing as in Part (ii), we have

$$0 = \Phi_c[\bar{u}_1] + \Phi_c[\bar{u}_2] = \Phi_c[\bar{u}_3] + \Phi_c[\bar{u}_4] \Rightarrow \Phi_c[\bar{u}_3] = \Phi_c[\bar{u}_4] = 0. \tag{4.44}$$

Therefore, $\bar{u}_3$ and $\bar{u}_4$ are also minimizers of Φ_c, and $w = \bar{u}_4 - \bar{u}_3 \geq 0$. Once again, using the arguments following (4.23) and taking into account that $w(x^*) = 0$, from Strong Maximum Principle we conclude that $w(x) = 0$ in all of Σ. So, $\bar{u}_1(y,z) = \bar{u}_2(y, z-a)$ for all $x = (y,z) \in \Sigma$.

This completes the proof of Theorem 4.3. □

Let us note that if the nonlinearity f is independent of y and the boundary conditions are Neumann, then the solution is essentially one-dimensional.

Proposition 4.4. *Let $\bar{u}$ be a solution obtained in Theorem 4.3, and assume that $\nabla_y f = 0$ and $\partial\Sigma_\pm = \varnothing$. Then $\bar{u}$ depends only on the variable z.*

Proof. The proof follows directly from the argument of Proposition 6.3 of [38]. □

In view of Proposition 4.4, the planar front solutions of Proposition 4.4 are also the fastest variational traveling waves among all waves with fixed y-independent nonlinearity and different choices of the boundary conditions.

The proof of Parts (iii), (iv) of Theorem 4.3 relied only on the fact that the minimizer is sandwiched between the two equilibria it connects. Using the same arguments as in Part (iii) of Theorem 4.3, it is also easy to show that for a variational traveling wave one should have $c^2 + 4\nu_0 > 0$ in order for the wave to have the right decay. So we have

Proposition 4.5. *Let $u_c \in H^1_c(\Sigma)$ be a solution of (4.1) which also satisfies $0 < u_c < v$, where $v = \lim_{z\to-\infty} u_c(\cdot, z)$ uniformly in Ω. Then, $c^2 + 4\nu_0 > 0$, and statement (iii) of Theorem 4.3 holds for u_c. If, in addition, $\tilde{\nu}_0 \neq 0$, statement (iv) of Theorem 4.3 holds for u_c as well.*

More generally, since, according to Proposition 4.5 and (4.11), any constant sign variational traveling wave is a trial function satisfying hypothesis (H3), the minimizer obtained in Theorem 4.3 is the fastest variational traveling wave. In other words, we have

Proposition 4.6. *If $u_c \in H^1_c(\Sigma)$ is a solution of (4.1), as in Proposition 4.5, then $c \leq c^\dagger$.*

In fact, the following stronger statement concerning *all* variational traveling waves that connect the same equilibria as the minimizer holds.

Proposition 4.7. *Let $u_c \in H^1_c(\Sigma)$ be a solution of (4.1), and let $0 < u_c < v$, where $v = \lim_{z\to-\infty} u_c(y,z)$ is the same as in Theorem 4.3, and $\tilde{\nu}_0 \neq 0$. Then $(c, u_c) = (c^\dagger, \bar{u})$, where the pair $(c^\dagger, \bar{u})$ is the solution obtained in Theorem 4.3.*

Proof. First of all, in view of Proposition 4.5 the fact that $u_c \in H^1_c(\Sigma)$ implies that u_c has the decay specified in Part (iii) of Theorem 4.3. By direct inspection, $\lambda_+(c, \nu_0) > 0$ and $\lambda_-(c, \tilde{\nu}_0) < 0$ are both increasing functions of c. Therefore, when $c < c^\dagger$, the solution u_c decays to zero slower exponentially than $\bar{u}$ as $z \to +\infty$, and faster to v as $z \to -\infty$. So, it is possible to translate $\bar{u}$ sufficiently far towards $z = -\infty$ to achieve $\bar{u} < u_c$ in Σ. Then, using Comparison Principle for parabolic equations [47] applied to the corresponding traveling wave solutions of (1.1), we see that u_c must move no slower than $c^\dagger$, which is impossible. Repeating this argument for $c > c^\dagger$, except now one has to translate $\bar{u}$ towards $z = +\infty$ to get an appropriate supersolution, we obtain a contradiction once more. □

In other words, there are no other variational traveling waves which are sandwiched between the same equilibria as the minimizer of $\Phi_{c^\dagger}$ obtained in Theorem 4.3. We also point out that under an assumption of non-degeneracy and uniqueness of the local minimizer $v_0 > 0$ with $E[v_0] < 0$ (which is then the global minimizer, see the discussion at the end of Section 3), the pair $(c^\dagger, \bar{u})$ from Theorem 4.3 is in fact the *only* variational traveling wave solution. In general, however, there may exist other variational traveling waves with speeds $c < c^\dagger$ which connect $u = 0$ to a local minimum of E other than v in Part (ii) of Theorem 4.3.

Remark 4.8. *In view of Proposition 4.5 and equation (4.11), existence of a minimizer necessarily implies that hypothesis (H3) is true. Thus, hypothesis (H3) is both necessary and sufficient for existence of variational traveling waves (this fact was already pointed out in [37] in the case $\Sigma = \mathbb{R}$).*

Now we would like to get back to considering an important special case $\nu_0 \geq 0$, i.e. the case when $u = 0$ is a locally stable (or marginally stable) solution of (1.1). Here, to satisfy hypothesis (H3) we just need to find a non-trivial trial function $u \in H^1_c(\Sigma)$ for which $\Phi_c[u] \leq 0$ for some small enough $c > 0$. In fact, the following stronger version of Theorem 4.3 holds.

Theorem 4.9. *Assume that hypotheses (H1) and (H2) hold, and that $\nu_0 \geq 0$ in (3.9). Then the statements of Theorem 4.3 remain true, if and only if*

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E[v] < 0. \tag{4.45}$$

Proof. The proof follows from a straightforward extension of the arguments of Proposition 6.2 of [38]. □

In other words, the statement of Theorem 4.9 holds if and only if there exists a non-trivial minimizer of E in the admissible class. In particular, if v_0 is the unique critical point with negative energy (necessarily the minimizer) and $\nu_0 > 0$, then there is a unique pair $(c^\dagger, \bar{u})$ solving (1.1). Indeed, by Proposition 4.7 the minimizer in Theorem 4.9 is the only variational traveling wave, and there are no other traveling wave solutions for $\nu_0 > 0$. Note that the same statement is true for the ignition-type nonlinearity from combustion theory under the assumption of uniqueness of v_0.

What if hypothesis (H3) is *not* satisfied? In this case, there are obviously no variational traveling wave solutions. However, if $\nu_0 < 0$, then it is possible to have traveling wave solutions which do not lie in $H^1_c(\Sigma)$. The following proposition gives a general sufficient condition for non-existence of minimizers for Φ_c and is a generalization of the earlier results of [37, 38].

Proposition 4.10. *Under hypotheses (H1) and (H2), assume that $\nu_0 < 0$ and*

$$\frac{2}{u^2}\int_0^u f(s,y)ds \leq f_u(0,y), \qquad \forall y \in \Omega. \tag{4.46}$$

Then the functional Φ_c has no non-trivial minimizers.

Proof. Let us first show that under this assumption $\Phi_c[u] = 0$ implies $u = 0$ for all $c \geq c_0$, where c_0 is given by (4.51). After an integration by parts, we can write

$$\Phi_c[u] = \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{1}{2}\left(u_z + \frac{c}{2}u\right)^2 + \frac{1}{2}|\nabla_y u|^2 + \frac{c^2}{8}u^2 + V(u,y) \right\} dx. \tag{4.47}$$

By the assumption of the proposition we have $V(u,y) \geq -\frac{1}{2} f_u(0,y)u^2$, and so

$$\begin{aligned} \Phi_c[u] &\geq \frac{1}{2}\int_\Sigma e^{cz+\varphi(y)} \left\{ \left(u_z + \frac{c}{2}u\right)^2 + |\nabla_y u|^2 + \left(\frac{c^2}{4} - f_u(0,y)\right) u^2 \right\} dx \\ &\geq \frac{1}{2}\int_\Sigma e^{cz+\varphi(y)} \left\{ \left(u_z + \frac{c}{2}u\right)^2 + \left(\frac{c^2}{4} + \nu_0\right) u^2 \right\} dx. \end{aligned} \tag{4.48}$$

The second term in the integrand above is non-negative for $c \geq c_0$, so $\Phi_c[u] = 0$ would imply that u is a minimizer and that the first term in the integrand is equal to zero. That, in turn, means that $u = v(y)e^{-cz/2}$ for some $v : \Omega \to \mathbb{R}$, and, in view of boundedness of the minimizers of Φ_c, we have $v \equiv 0$.

Let us now show that when $c < c_0$, the functional Φ_c is not bounded from below. For that, consider a trial function

$$u_\lambda(y,z) = \begin{cases} a\psi_0(y)e^{-\lambda z}, & z > 0, \\ a\psi_0(y), & z \leq 0, \end{cases} \tag{4.49}$$

where $\psi_0 > 0$ is the zeroth eigenfunction of the operator in (4.4) and $\lambda > \frac{c}{2}$. Choosing $a > 0$ small enough, we can always make $V(u_\lambda(y,z),y) \leq -\frac{1}{2}(f_u(0,y)-$

$\varepsilon)u_\lambda^2$ for any $\varepsilon > 0$. Plugging u_λ into the functional, we obtain

$$\begin{aligned}
\Phi_c[u_\lambda] &\le \frac{1}{2}\int_{-\infty}^{0}\int_\Omega e^{cz+\varphi(y)}\{|\nabla_y u_\lambda|^2 - (f_u(0,y)-\varepsilon)u_\lambda^2\}\,dydz \\
&\quad + \frac{1}{2}\int_{0}^{+\infty}\int_\Omega e^{cz+\varphi(y)}\{|\nabla_y u_\lambda|^2 + (\lambda^2+\varepsilon - f_u(0,y))u_\lambda^2\}\,dydz \\
&= \frac{a^2}{2c}\int_\Omega e^{\varphi(y)}(|\nabla\psi_0|^2 - (f_u(0,y)-\varepsilon)\psi_0^2)\,dy \\
&\quad + \frac{a^2}{2(2\lambda - c)}\int_\Omega e^{\varphi(y)}(|\nabla_y\psi_0|^2 + (\lambda^2+\varepsilon - f_u(0,y))\psi_0^2)\,dy \\
&= \frac{a^2}{2}\left(\frac{\nu_0+\varepsilon}{c} + \frac{\lambda^2+\varepsilon+\nu_0}{2\lambda - c}\right)\int_\Omega e^{\varphi(y)}\psi_0^2\,dy. \qquad (4.50)
\end{aligned}$$

It is then easy to see that the last line in the expression above can be made arbitrarily large negative for sufficiently small ε and $c < c_0$ by choosing λ sufficiently close to $\frac{c}{2}$. Therefore, in this case the functional Φ_c has no minimizers. □

A typical example of the situation in which Φ_c has no non-trivial minimizers is the KPP-type nonlinearity, e.g. $f(u) = u(1-u)$. Non-existence of variational traveling waves in this case follows from the above Proposition. However, our variational procedure allows us to establish existence of an important class of traveling wave solutions, having the speed which is equal to the minimal speed $c = c_0$, where c_0 is defined in (4.51), allowed for a positive traveling wave solution. As in the case of variational traveling waves, this solution turns out to determine the asymptotic propagation speed for the initial data that are sufficiently localized (see Sec. 7). Thus, to summarize the results of Theorem 4.3 and Theorem 4.11 below, under the condition in (4.45) there always exists a positive monotone traveling wave solution with speed satisfying $c^2 + 4\nu_0 \ge 0$ which decays exponentially at $z = +\infty$. This generalizes the well-known results of [12].

Theorem 4.11. *Assume that hypotheses (H1) and (H2) hold, whereas hypothesis (H3) is not satisfied. Assume in addition that $\nu_0 < 0$. Then, there exists $u_0 \in C^2(\Sigma) \cap W^{1,\infty}(\Sigma)$ which solves (4.1) with $c = c_0$, where*

$$c_0 = 2\sqrt{-\nu_0}. \qquad (4.51)$$

Furthermore, u_0 has the limiting behavior

$$u_0(y,z) = (a_0 + b_0 z)\,e^{-\frac{1}{2}c_0 z}\psi_0(y) + O(e^{-\lambda z}), \qquad (4.52)$$

for some $\lambda > \frac{c_0}{2}$ and either $b_0 > 0$ or $b_0 = 0, a_0 > 0$, as $z \to +\infty$, and assertions (ii) and (iv) of Theorem 4.3 still hold for u_0.

Proof. We prove this theorem by approximating the solution (c_0, u_0) of (4.1) with pairs $(c_\varepsilon, u_\varepsilon)$ solving

$$\Delta u_\varepsilon + c_\varepsilon \frac{\partial u_\varepsilon}{\partial z} + \nabla_y\varphi\cdot\nabla_y u_\varepsilon + f_\varepsilon(u_\varepsilon, y) = 0, \qquad (4.53)$$

$$u_\varepsilon\big|_{\partial\Sigma_\pm} = 0, \qquad \nu\cdot\nabla u_\varepsilon\big|_{\partial\Sigma_0} = 0, \qquad (4.54)$$

where

$$f_\varepsilon(u,y) = f(u,y) - \varepsilon K \tanh(u/\varepsilon), \qquad K = \max_{y\in\overline{\Omega}} f_u(0,y). \tag{4.55}$$

Associated with f_ε are the function V_ε and the functionals Φ_c^ε, E_ε, and R_ε, defined with f_ε in place of f. Note that by the definition of f_ε and hypothesis (H2), we have

$$0 \le f(u,y) - f_\varepsilon(u,y) \le \varepsilon K, \qquad \forall u \in [0,1], \quad \forall y \in \Omega, \tag{4.56}$$

and $f_\varepsilon(u,y)$ is a monotonically decreasing function of ε.

Observe that the assumption $\nu_0 < 0$ implies (4.45), since for $a > 0$ sufficiently small we have

$$\inf E \le E[a\psi_0] = \tfrac{1}{2}\nu_0 a^2 + o(a^2) < 0. \tag{4.57}$$

So, by continuity $\inf E_\varepsilon < 0$ for sufficiently small ε. We also have $\left.\frac{\partial f_\varepsilon(u,y)}{\partial u}\right|_{u=0} \le 0$ for all $y \in \Omega$. Hence $\nu_0^\varepsilon \ge 0$, where ν_0^ε is defined as the minimum of R_ε. Then, by Theorem 4.9 there exists a pair $(c_\varepsilon, u_\varepsilon)$, with $u_\varepsilon \in H^1_{c_\varepsilon}(\Sigma)$, which is the minimizer of $\Phi^\varepsilon_{c_\varepsilon}$, with all the properties given by Theorem 4.3. In particular, we have

$$\lim_{z\to-\infty} u_\varepsilon(y,z) = v_\varepsilon(y) \quad \text{in } C^1(\overline{\Omega}), \tag{4.58}$$

$$\Delta_y v_\varepsilon + \nabla_y \varphi \cdot \nabla v_\varepsilon + f_\varepsilon(v_\varepsilon, y) = 0, \quad E_\varepsilon[v_\varepsilon] < 0. \tag{4.59}$$

From the definition of f_ε and the Maximum Principle, we have $0 < v_\varepsilon \le 1$ for all ε, and by assumption $|\nabla_y \varphi| \in L^\infty(\Omega)$. Hence, $f(v_\varepsilon, y)$ are equibounded in $L^p(\Omega)$ with any p, and so v_ε are equibounded in $W^{2,p}(\Omega)$. Choosing p large enough and an appropriate sequence, still labeled v_ε, we then extract a limit $v_0 = \lim v_\varepsilon$ in $C^1(\overline{\Omega})$. Since, according to (4.56) we have $f_\varepsilon \to f$ uniformly in Ω, as $\varepsilon \to 0$, and f is continuous, we obtain that v_0 is a (weak) solution of (3.11). By standard regularity arguments, v_0 is also a classical solution. Also, by monotonicity of f_ε we have that v_ε is an increasing sequence of functions, and so $v_0 > 0$ in Ω.

From here on, we restrict all the arguments to the sequence of $\varepsilon \to 0$ constructed above. Observe that in view of monotonicity of V_ε as a function of ε the sequence of c_ε is monotone increasing. Furthermore, it is bounded from above by c_0. Indeed, $\Phi_{c_\varepsilon}[u_\varepsilon] \le \Phi^\varepsilon_{c_\varepsilon}[u_\varepsilon] = 0$, and if $c_\varepsilon > c_0$, then the pair $(c_\varepsilon, u_\varepsilon)$ satisfies hypothesis (H3) for the original problem, which is false. So, $c_\varepsilon \le c_0$. Let us show that in fact $c_0 = \lim c_\varepsilon$. Indeed, fix a sufficiently small $a > 0$ and consider a trial function

$$\tilde{u}_\varepsilon(y,z) = \begin{cases} a\psi_0(y), & z < 0, \\ ae^{-cz/2}\psi_0(y), & 0 \le z \le R, \\ ae^{-cR/2}\left(1 - \frac{c(z-R)}{2}\right)\psi_0(y), & R \le z \le R + \frac{2}{c}, \\ 0, & z > R + \frac{2}{c}. \end{cases} \tag{4.60}$$

for some $R > 0$. By construction, $\tilde{u}_\varepsilon \in H^1_c(\Sigma)$. Now, introduce

$$\Omega_\varepsilon = \{y \in \Omega : a\psi_0(y)e^{-cR/2} < \varepsilon\} \tag{4.61}$$

This set is non-empty for small enough ε, and, furthermore, $|\Omega_\varepsilon| \to 0$ for $\varepsilon \to 0$. Then

$$\begin{aligned}
\Phi^\varepsilon_c[\tilde{u}_\varepsilon] &= \int_{-\infty}^0 \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz \\
&\quad + \int_0^{+\infty} \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz \\
&\leq \frac{1}{c}\int_\Omega e^{\varphi(y)} \left(\frac{a^2}{2}|\nabla_y \psi_0|^2 + V_\varepsilon(a\psi_0(y), y)\right) dy \\
&\quad + \int_0^R \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V(\tilde{u}_\varepsilon, y)\right) dy\, dz \\
&\quad + \int_0^R \int_{\Omega_\varepsilon} e^{cz+\varphi(y)} \left(V_\varepsilon(\tilde{u}_\varepsilon, y) - V(\tilde{u}_\varepsilon, y)\right) dy\, dz \\
&\quad + \int_R^{R+\frac{2}{c}} \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz.
\end{aligned} \tag{4.62}$$

By hypothesis (H2), it is possible to choose the constant a such that

$$V(\tilde{u}_\varepsilon, y) \leq -\tfrac{1}{2}\left(f_u(0, y) - \delta\right)\tilde{u}_\varepsilon^2, \qquad \forall \delta > 0. \tag{4.63}$$

Also, we have a uniform estimate $|V_\varepsilon(u, y)| \leq Cu^2/2$, where C is independent of ε or y. Therefore, continuing the argument above, we can write

$$\begin{aligned}
\Phi^\varepsilon_c[\tilde{u}_\varepsilon] &\leq \frac{a^2}{2c}\int_\Omega e^{\varphi(y)} \left(|\nabla_y \psi_0|^2 + C\psi_0^2\right) dy \\
&\quad + \frac{a^2}{2}\int_0^R \int_\Omega e^{\varphi(y)} \left(\frac{c^2}{4}\psi_0^2 + |\nabla_y \psi_0|^2 - f_u(0, y)\psi_0^2 + \delta\psi_0^2\right) dy\, dz \\
&\quad + Ca^2 \int_0^R \int_{\Omega_\varepsilon} e^{\varphi(y)} \psi_0^2 dy\, dz \\
&\quad + \frac{a^2}{2}\int_R^{R+\frac{2}{c}} \int_\Omega e^{c(z-R)+\varphi(y)} \left(\frac{c^2}{4}\psi_0^2 + |\nabla_y \psi_0|^2 + C\psi_0^2\right) dy\, dz. \\
&\leq \frac{a^2}{2c}\int_\Omega e^{\varphi(y)} \left(|\nabla_y \psi_0|^2 + C\psi_0^2\right) dy \\
&\quad + \frac{1}{8}Ra^2(c^2 + 4\nu_0 - \delta)\int_\Omega e^{\varphi(y)}\psi_0^2\, dy + Ca^2 R|\Omega_\varepsilon| \\
&\quad + \frac{9a^2}{c}\int_\Omega e^{\varphi(y)} \left(\frac{c^2}{4}\psi_0^2 + |\nabla_y \psi_0|^2 + C\psi_0^2\right) dy \\
&= a^2 M_1 R(c^2 + 4\nu_0 + \delta + M_2|\Omega_\varepsilon|) + a^2 M_3,
\end{aligned} \tag{4.64}$$

where M's are positive constants independent of $\varepsilon, \delta, a, R$.

Now, for any positive $c < c_0$ it is possible to choose $\delta > 0$ small enough such that $c^2 + 4\nu_0 + \delta < 0$. This fixes the value of a. Then, choose R large enough, so that $M_1 R(c^2 + 4\nu_0 + \delta) + M_3 < 0$. Then, since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$, there exists $\varepsilon > 0$ such that $\Phi_c^\varepsilon[\tilde{u}_\varepsilon] \leq 0$, and so $c_\varepsilon \geq c$ for some small enough ε. In view of arbitrariness of c, this implies $c_\varepsilon \to c_0$.

Now we construct the limit function u_0 and show that it satisfies (4.1) with $c = c_0$. Notice that, recalling Remark 4.2, the assumption $\nu_0 < 0$ implies that there exists $\delta > 0$ such that 0 is the only critical point of E taking values in $[0, \delta]$. In particular, we have $\max_{\overline{\Omega}} v_0 > \delta$. Recalling the monotonicity and the limit behavior of u_ε, as $z \to \pm\infty$, after an appropriate translation we can assume that u_ε satisfies

$$\max_{y \in \overline{\Omega}} u_\varepsilon(y, 0) = \delta \tag{4.65}$$

for small enough ε. As the functions u_ε are uniformly bounded in $W^{1,\infty}(\Sigma)$ and in $W^{2,p}_{\text{loc}}(\Sigma)$, we can pass to the limit as $\varepsilon \to 0$, and obtain a function $u_0 \in C^2(\Sigma) \cap W^{1,\infty}(\Sigma)$ which solves (4.1) with speed $c = c_0$. Moreover, we have $0 \leq u_0 \leq 1$, and u_0 satisfies (4.65) and, hence, is not identically zero. Furthermore, u_0 is non-increasing in the z-variable, and so by Strong Maximum Principle we have $0 < u_0 < 1$ and $\partial u_0 / \partial z < 0$ in Σ.

Reasoning as in the proof of [38, Proposition 6.6], we can show that u_0 connects two critical points $v_\pm$ of E, for $z \to \pm\infty$ respectively, with $0 \leq v_+ \leq \delta$ and $E[v_-] < E[v_+]$. By (4.65) and Remark 4.2 it follows that $v_+ = 0$ and $v_- = v$, where $0 < v \leq v_0$, and hence $E[v] < 0$. Notice that we could have $v \neq v_0$.

The asymptotic expansion in (4.52) follows from exactly the same arguments as in the proof of Part (iii) of Theorem 4.3. The only ingredient that is missing here is an a priori estimate of exponential decay of the solution (since it may no longer lie in any of the exponentially weighted Sobolev spaces $H^1_c(\Sigma)$). To overcome this difficulty, consider (4.31) with $k = 0$:

$$a_0'' + c_0 a_0' + \tfrac{1}{4} c_0^2 a_0 = g_0. \tag{4.66}$$

By the same argument as the one leading to (4.41), we have $|g_0(z)| \leq \varepsilon a_0(z)$ with arbitrary $\varepsilon > 0$ when z is large enough. Therefore, it is easy to see that $\bar{a}_0(z) = a_0(R) e^{-\lambda(z-R)}$ with $\lambda = \frac{1}{2} c_0 - \sqrt{\varepsilon}$ is a supersolution for large enough R, implying that $a_0(z)$ decays exponentially to zero as $z \to +\infty$.

This, in turn, implies exponential decay of u_0. Indeed, let $u_m(z) = \max_{y \in \overline{\Omega}} u_0(y, z)$ and $y_m(z)$ the location of this maximum in $\overline{\Omega}$. By previous results we have $u_m \to 0$ as $z \to +\infty$. Now, by regularity of $\partial\Omega$, there exists a closed cone $\mathcal{C}_\Omega$ (with finite height) such that each point $y \in \overline{\Omega}$ is a vertex of a cone $\mathcal{C}_y \subset \overline{\Omega}$ congruent to $\mathcal{C}_\Omega$. Therefore, by the uniform estimate on $|\nabla u_0|$ for each z sufficiently large there exists a cone $\tilde{\mathcal{C}}_{y_m} \subseteq \mathcal{C}_{y_m}$ similar to $\mathcal{C}_{y_m}$ such that $u_0(y, z) \geq \frac{1}{2} u_m(z)$ for all $y \in \tilde{\mathcal{C}}_{y_m}$ and $|\tilde{\mathcal{C}}_{y_m}| = C_1 u_m^{n-1}(z)$, with some $C_1 > 0$ independent of z. Also, since $\text{dist}(\tilde{\mathcal{C}}_{y_m}, \partial\Omega_\pm) \geq C_2 u_m$ and by Hopf Lemma the normal derivative of

ψ_0 on $\partial\Omega_\pm$ is bounded from below [30, Lemma 3.4], we also have $\psi_0(y) \geq C_3 u_m$ for all $y \in \mathcal{C}_{y_m}$. Using these estimates in the definition of a_0, we get

$$a_0(z) = \int_\Omega e^{\varphi(y)} u_0(y,z)\psi_0(y)dy \geq \int_{\tilde{\mathcal{C}}_{y_m}} e^{\varphi(y)} u_0(y,z)\psi_0(y)dy \geq C u_m^{n+1}, \quad (4.67)$$

and so $u_m \leq C a_0^{\frac{1}{n+1}}(z) \leq Ce^{-\mu z}$, with some $\mu > 0$, for large enough z. □

Note that in general there is no uniqueness in Theorem 4.11. This can be easily seen from the phase plane analysis already in the case $\Sigma = \mathbb{R}$ in the presence of multiple equilibria. On the other hand, under an extra assumption of non-degeneracy of $v = \lim_{z\to-\infty} u(\cdot, z)$, uniqueness follows from the sliding domain method of Berestycki and Nirenberg [11] in the class of solutions with the same limit at $z = -\infty$.

Remark 4.12. *As follows from the argument of Kawohl [34], if $s \mapsto V(\sqrt{s}, y)$ is a strictly convex function of s for any fixed $y \in \Omega$, there may only exist a unique critical point (necessarily a minimizer) of E with negative energy. Since the assumption of convexity above implies the condition in Proposition 4.10, we are automatically dealing with the case covered by Theorem 4.11.*

We illustrate this situation with an example of the Allen-Cahn equation, for which $f(u) = u(1-u^2)$. Letting $w = v^2$, we can rewrite the functional E as

$$E[v] = \widetilde{E}[w] = \int_\Omega e^{\varphi(y)} \left(\frac{|\nabla w|^2}{8w} - \frac{w}{2} + \frac{w^2}{4} \right) dy. \quad (4.68)$$

By inspection, $\widetilde{E}$ is strictly convex on $w > 0$ (which corresponds to $v > 0$), and so it admits at most one critical point with negative energy. If it does, there exists a unique (up to translations) traveling wave solution of (4.1) with speed c_0. Let us also point out that the classical Fisher nonlinearity $f(u) = u(1-u)$ satisfies the assumption of Remark 4.12.

5 Sharp reaction zone limit

In this section, we consider a singular limit of (4.1) which arises in the context of combustion theory. Namely, we consider the situation in which the nonlinearity f is independent of y and set formally $f = f_0$, where $f_0(u) = \delta(u - 1^-)$, and $\delta(x)$ is the Dirac delta-function (see Sec. 2). This choice of f is meant to approximate a rapidly increasing reaction rate as a function of temperature due to the Arrhenius law, see (2.12), and gives a situation where all of the reaction occurs at a critical value $u = 1$. We note, however, that in general it is difficult to assign a meaning to (1.1) or (4.1) with such a singular nonlinearity. Nevertheless, due to the localized character of the reaction it is possible to give a satisfactory interpretation for these equations in terms of a free boundary problem in which the reaction zone is described as a two-dimensional surface (in the physical case of $n = 3$) separating the so-called "preheat zone" from the "products" in combustion terminology [2, 7, 55].

On the other hand, it is possible to pass to the limit of $f = f_0$ directly in (3.3). Introducing

$$V_0(u) = \begin{cases} 0 & u < 1, \\ -1 & u \geq 1, \end{cases} \tag{5.1}$$

we see that $V_\varepsilon \to V_0$ as $\varepsilon \to 0$, if V_ε is given by (3.4) with $f_\varepsilon \to \delta(u - 1^-)$. We note that V_0 defined in this way is lower semicontinuous, making further variational analysis of the problem possible.

Setting $V = V_0$ in (3.3), we introduce the functional

$$\Phi_c^0[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2} |\nabla u|^2 + V_0(u) \right) dx. \tag{5.2}$$

The results of the previous section motivate us to analyze the minimizers of (5.2). Our main result here is a characterization of the uniformly translating solutions of the free boundary problem associated with (1.1) with the minimizers of (5.2). As in Theorem 4.9, existence of the minimizers can be established in terms of the auxiliary functional $E_0[v]$, defined for all $v \in H^1(\Omega)$ that vanish on $\partial\Omega_\pm$, which for $V = V_0$ can be written simply as

$$E_0[v] = \frac{1}{2} \int_{\{v<1\}} e^{\varphi(y)} |\nabla v|^2 \, dy - \int_{\{v \geq 1\}} e^{\varphi(y)} \, dy. \tag{5.3}$$

We point out that a functional of this kind has been considered in [2]. Later, in Sec. 6, we prove that these minimizers are in fact limits of the corresponding approximation problems in (2.16).

Here is our main result concerning the minimizers of Φ_c^0.

Theorem 5.1. *Let $n \leq 3$, and assume that there exists $c > 0$ and $u \in H_c^1(\Sigma)$, $u \not\equiv 0$, such that $\Phi_c^0[u] \leq 0$. Then:*

(i) There exists a unique constant $c^\dagger \geq c$ and $\bar{u} \in H^1_{c^\dagger}(\Sigma) \cap W^{1,\infty}(\overline{\Sigma})$, $\bar{u} \not\equiv 0$, such that $\bar{u}$ is a minimizer of $\Phi^0_{c^\dagger}$. Moreover, $0 < \bar{u} \leq 1$ in Σ, and if

$$\omega_- = \{x \in \Sigma : \bar{u}(x) < 1\}, \qquad \omega_+ = \{x \in \Sigma : \bar{u}(x) = 1\}, \tag{5.4}$$

then the set ω_+ is non-empty, and $\bar{u}$ is a classical solution of

$$\Delta \bar{u} + c\bar{u}_z + \nabla_y \varphi \cdot \nabla \bar{u} = 0, \qquad \bar{u}\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla \bar{u}\big|_{\partial\Sigma_0} = 0, \tag{5.5}$$

in ω_-.

(ii) The function $\bar{u}(y, z)$ is unique (up to translations), is strictly decreasing in z in ω_-, and $\lim_{z\to+\infty} \bar{u}(\cdot, z) = 0$ in $C^1(\overline{\Omega})$.

(iii) $\bar{u}(y,z) = a_0\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)z} + O(e^{-\lambda z})$, *with some* $a_0 > 0$ *and* $\lambda > \lambda_+(c^\dagger,\nu_0)$, *uniformly in* $C^1(\overline{\Omega} \times [R,+\infty))$ *as* $R \to +\infty$, *where* $\psi_0 > 0$ *and* $\lambda_+(c^\dagger,\nu_0)$ *are defined as in* (4.4) *and* (4.8), *with* $f \equiv 0$.

(iv) The free boundary $\partial\omega_\pm = \partial\omega_- \cap \partial\omega_+$ *is bounded from the right and has regularity* $C^{1,\alpha}$, *for some* $\alpha > 0$. *Moreover* $\bar{u} \in C^{1,\alpha}(\overline{\omega}_-)$, *and the following boundary condition holds:*

$$\bar{u}\,|_{\partial\omega_\pm} = 1, \qquad \nu \cdot \nabla\bar{u}\,|_{\partial\omega_\pm} = -\sqrt{2}, \tag{5.6}$$

where ν *is the normal to* $\partial\omega_\pm$ *pointing into* ω_-.

(v) $\lim_{z\to-\infty} \bar{u}(\cdot,z) = v$ *uniformly in* Ω, *where* $v > 0$ *is a critical point of* E_0 *such that* $E_0[v] < 0$. *In particular, letting* $\omega_R = \omega_+ \cap \{z = R\}$ *and*

$$\omega_0 = \bigcup_{R\in\mathbb{R}} \omega_R \subseteq \Omega, \tag{5.7}$$

we have $\partial\omega_R \to \partial\omega_0$ *in the Hausdorff sense as* $R \to -\infty$, $\partial\omega_0$ *is of class* $C^{1,\alpha}$, *for some* $\alpha > 0$, *and* $\partial\omega_\pm$ *is a graph of a function* $h \in BV_{\mathrm{loc}}(\omega_0)$.

Proof of Part (i)

Existence of a minimizer, uniqueness of $c^\dagger$, and the fact that $\bar{u}(x) \in [0,1]$ for all $x \in \Sigma$ follow from the same arguments as in Theorem 4.3, Part (i). The only difference is in the proof of the inclusion $\bar{u} \in W^{1,\infty}(\overline{\Sigma})$. The latter follows from the fact that $\bar{u}$ is a minimizer of $\Phi^0_{c^\dagger}$, reasoning as in [2, Corollary 3.3] (see also [7, Theorem 3.1]).

Let us rewrite the functional Φ^0_c as

$$\Phi^0_c[\bar{u}] = \frac{1}{2}\int_{\omega_-} e^{cz+\varphi(y)}|\nabla\bar{u}|^2 dx - \int_{\omega_+} e^{cz+\varphi(y)} dx. \tag{5.8}$$

In view of Lemma 3.2, if $\omega_+ = \varnothing$, then $\Phi^0_{c^\dagger}[\bar{u}] \geq 0$ and $\Phi^0_{c^\dagger}[\bar{u}] = 0$ if and only if $\bar{u} = 0$, contradicting the existence of a nontrivial minimizer. Hence $\omega_+ \neq \varnothing$.

Since $V_0 = 0$ in ω_-, the Gateaux derivative $D_\phi\Phi^0_{c^\dagger}[\bar{u}]$ exists and is equal to zero for all test functions ϕ vanishing on ω_+. So $\bar{u}$ solves the Euler-Lagrange equation for $\Phi^0_{c^\dagger}$ with $V_0 = 0$ in ω_-, which is precisely equation (5.5). Then, by the Strong Maximum Principle, we have $\bar{u} > 0$ in ω_- and, therefore, in the whole of Σ.

Proof of Part (ii)

Uniqueness, monotonicity and uniform decay of $\bar{u}$ follow reasoning as in Theorem 4.3, Parts (ii) and (v). We only point out a few modifications of the arguments above. Letting $\bar{u}_1, \bar{u}_2$ be as in Part (ii) of Theorem 4.3, for a given translation $a > 0$, we have $\Phi_{c^\dagger}[\bar{u}_1] = \Phi_{c^\dagger}[\bar{u}_2] = 0$, hence $\bar{u}_1, \bar{u}_2$ are non-trivial minimizers of $\Phi^0_{c^\dagger}$. It follows that the difference $w = \bar{u}_2 - \bar{u}_1 \geq 0$ solves the equation

$$\Delta w + c^\dagger w_z + \nabla_y \varphi \cdot \nabla_y w = 0 \qquad \text{in the set} \qquad \{\bar{u}_2 < 1\}\,. \tag{5.9}$$

It follows that either $w = 0$ or $w > 0$ in $\{\bar{u}_2 < 1\}$. The first possibility would imply that $\bar{u}$ is independent of z and, hence, is zero, which is impossible. So, $w > 0$, implying that $\bar{u}(y, z - a) > \bar{u}(y, z)$ for all (y, z) such that $\bar{u}_2(y, z) < 1$. In view of the arbitrariness of $a > 0$, this implies that $\bar{u}$ is strictly monotone decreasing in ω_-.

Similarly, let $\bar{u}_3$ and $\bar{u}_4$ be as in Part (v) of Theorem 4.3, with $\bar{u}_3(x^*) = \bar{u}_4(x^*) < 1$. Then, since $w = \bar{u}_4 - \bar{u}_3$ satisfies (5.9) in the set $\{\bar{u}_4 < 1\}$, with $w \geq 0$ on the boundary of $\{\bar{u}_4 = 1\}$, from the Strong Maximum Principle we conclude that $w \equiv 0$ (hence also on the boundary). Then, by monotonicity of the minimizers, $\bar{u}_3$ and $\bar{u}_4$ coincide in all of Σ.

Proof of Part (iii)

This is just a particular case of Theorem 4.3, Part (iii), with $f \equiv 0$ for z large enough. Notice that in this case $\nu_0 \geq 0$ and $\lambda_+(c^\dagger, \nu_0) \geq c^\dagger$.

Proof of Part (iv)

Notice first that $\partial\omega_\pm$ is bounded from the right since $\bar{u}(\cdot, z) \to 0$ uniformly, by Part (ii). The fact that $\bar{u} \in C^{1,\alpha}(\overline{\omega}_-)$ and $\partial\omega_\pm$ is of class $C^{1,\alpha}$, for some $\alpha > 0$, follows from [2, 18]. Here we use that $n \leq 3$, since otherwise the boundary set $\partial\omega_\pm$ could contain singular points [21]. Indeed, reasoning as in the proof of [45, Theorem 1.2], we have that $\partial\omega_\pm$ is *uniformly* of class $C^{1,\alpha}$, in the sense that there exists $\rho, C > 0$ such that $\partial\omega_\pm \cap B_\rho(x)$ is contained in the graph (along some direction) of a function with $C^{1,\alpha}$-norm bounded by C, for all $x \in \partial\omega_\pm$.

The free boundary condition in (5.6) is also obtained in [2]. For reader's convenience we present the argument here. The first condition follows from the definition of $\partial\omega_\pm$ and the continuity of $\bar{u}$ established in Part (i). Let us prove the second condition in (5.6). Fix $\varepsilon > 0$ and a function $\rho \in C^{1,\alpha}(\partial\omega_\pm)$. We perturb $\partial\omega_\pm$ by displacing each point of $\partial\omega_\pm$ by $\varepsilon\rho \leq 0$ along ν, where ν is the normal to $\partial\omega_\pm$ pointing into ω_-. In order to preserve the first boundary condition in (5.6), we also perturb the function $\bar{u}$ by adding to it $\varepsilon\phi$, where the function ϕ satisfies the same boundary conditions as $\bar{u}$ on $\partial\Sigma$ and solves in ω_- the following boundary value problem:

$$\Delta\phi + c\phi_z + \nabla_y \varphi \cdot \nabla_y \phi = 0, \qquad \phi\Big|_{\partial\omega_\pm} = -(\nu \cdot \nabla\bar{u})\,\rho + o(\varepsilon). \tag{5.10}$$

The derivative of Φ^0_c with respect to ε becomes

$$\begin{aligned} 0 = \frac{d\Phi^0_c[\bar{u}]}{d\varepsilon} &= -\frac{1}{2}\int_{\partial\omega_\pm} e^{cz+\varphi(y)} |\nabla\bar{u}|^2 \rho\, d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial\omega_\pm} e^{cz+\varphi(y)} \rho\, d\mathcal{H}^{n-1} + \int_{\omega_-} e^{cz+\varphi(y)} \nabla\bar{u} \cdot \nabla\phi\, dx. \end{aligned} \tag{5.11}$$

Integrating by parts and noting that on $\partial\omega_\pm$ we have $\nu \cdot \nabla \bar{u} = -|\nabla \bar{u}|$, after some algebra we obtain

$$0 = \frac{d\Phi_c^0[\bar{u}]}{d\varepsilon} = \frac{1}{2} \int_{\partial\omega_\pm} e^{cz+\varphi(y)} \left(|\nabla \bar{u}|^2 - 2\right) \rho \, d\mathcal{H}^{n-1}. \tag{5.12}$$

Therefore, the following condition defines the location of the free boundary $\partial\omega_\pm$:

$$|\nabla \bar{u}| = \sqrt{2} \quad \text{on} \quad \partial\omega_\pm. \tag{5.13}$$

In view of the fact that $\bar{u}$ decreases along ν, we obtain the statement.

Proof of Part (v)

The existence of a function $v \in W^{1,\infty}(\Omega)$, such that $\lim_{z\to-\infty} \bar{u}(\cdot, z) = v > 0$ uniformly in Ω, follows from the Lipschitz continuity and monotonicity of $\bar{u}$, proved in Parts (i) and (ii) respectively. Notice also that $v \equiv 1$ in $\overline{\omega}_0$.

Since $\omega_R \subseteq \omega_0$ for all $R \in \mathbb{R}$ and $|\omega_0 \setminus \omega_R| \to 0$, as $R \to -\infty$, the Hausdorff convergence of $\partial\omega_R$ to $\partial\omega_0$ follows from the fact that $\partial\omega_R$ are uniformly of class $C^{1,\alpha}$, independently of R, as stated in the proof of Part (iv). It then follows that $\partial\omega_0$ is also of class $C^{1,\alpha}$.

We now show that the function $\bar{v}(y, z) = v(y)$ is a minimizer for Φ_c^0 on Σ, with respect to perturbations with *bounded* support. Indeed, letting $a, b, R \in \mathbb{R}$, with $a < b$, the function $\bar{u}_R(y, z) = \bar{u}(y, z - R)$ is a minimizer for Φ_c^0 restricted to $\Sigma_{a,b} = \Omega \times (a, b)$, with respect to perturbations vanishing on $\partial\Sigma_{a,b} \setminus \left(\partial\Omega_0 \times (a, b)\right)$. Since $\|\bar{u}_R - \bar{v}\|_{L^\infty(\Sigma_{a,b})} \to 0$ as $R \to -\infty$, it follows that $\bar{v}$ is also a minimizer for Φ_c^0 restricted to $\Sigma_{a,b}$, with respect to such perturbations. In particular, since $\bar{u}_R$ satisfies equation (5.5) in $\left(\Omega \setminus \overline{\omega}_0\right) \times \mathbb{R}$, we obtain that v solves the linear equation (5.14) in $\{v < 1\}$. In particular $v \in (0, 1)$ in $\Omega \setminus \overline{\omega}_0$, by Strong Maximum Principle. Moreover, arguing as in Part (iv), we get that v satisfies the boundary condition (5.15). Equations (5.14) and (5.15) imply that v is a critical point of E_0. The inequality $E_0[v] < 0$ follows as in Theorem 4.3, Part (ii).

Finally, since $\partial\omega_\pm$ has locally finite perimeter in Σ (being of class $C^{1,\alpha}$) and $\bar{u}$ is monotone in the z-direction, we have that $\partial\omega_\pm$ is a graph of a function $h \in BV_{\mathrm{loc}}(\omega_0)$. □

Notice that, while Theorem 5.1 covers the physically relevant case $n \leq 3$, most of its statements can be extended to arbitrary dimensions. The only difficulty in $n \geq 4$ is the lack of complete regularity theory for the free boundary $\partial\omega_\pm$ [2, 21]. It is currently known that the free boundary is regular, out of possibly a closed singular set $S_\pm \subset \partial\omega_\pm$ of Hausdorff dimension at most $n - 4$ [61]. We note that, since in our case, the free boundary is a graph in the z-direction, we expect that the singular set be always empty, independently of the dimension [18, 61].

Remark 5.2. *The set $\overline{\omega}_0$ in Part (v) of Theorem 5.1 is the set on which $v = 1$, and*

$$\Delta_y v + \nabla_y \varphi \cdot \nabla_y v = 0, \qquad v|_{\partial\Omega_\pm} = 0, \quad \nu \cdot \nabla_y v|_{\partial\Omega_0} = 0 \tag{5.14}$$

in $\Omega \backslash \overline{\omega}_0$, and the free boundary conditions

$$v|_{\partial\omega_0 \backslash \partial\Omega} = 1, \qquad \nu \cdot \nabla_y v|_{\partial\omega_0 \backslash \partial\Omega} = -\sqrt{2}, \tag{5.15}$$

where ν is the normal to $\partial\omega_0$ pointing outside ω_0.

Arguing as in Theorem 4.9, we obtain the following necessary and sufficient condition for the considered problem to have minimizers:

Corollary 5.3. *Minimizers of Φ_c^0 exist if and only if*

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E_0[v] < 0. \tag{5.16}$$

We note that, conversely, existence of a solution of the free boundary problem in Theorem 5.1 implies existence of minimizers of Φ_c^0. Indeed, if u_c is a solution of the free boundary problem, it satisfies a linear equation (5.5) in ω_- and is given by the series in (4.2). Furthermore, we have $\lambda_k = \lambda_+(c, \nu_k) > \frac{c}{2}$, since $\nu_k \geq 0$, for all $k = 0, 1, \ldots$, in view of $f_u(0, y) = 0$ in (4.4). So $u_c \in H_c^1(\Sigma)$, and repeating the arguments in the proof of Theorem 5.1, we conclude that u_c is a critical point of Φ_c^0. This, in turn, implies that $\Phi_c^0[u_c] = 0$, and so u_c can be used as a trial function in the assumptions of Theorem 5.1. Thus, non-existence of minimizers in Corollary 5.3 implies non-existence of solutions of the free boundary problem as well.

Let us also point out that the statement in part (v) of Theorem 5.1 includes the possibility that the free boundary has "vertical" portions (i.e. those with $\nu \cdot \hat{z} = 0$). However, one would expect that generally $\partial\omega_\pm$ does not have any such portions and thus is a graph of a $C^{1,\alpha}_{\mathrm{loc}}(\omega_0)$ function. In fact, when $n = 2$, it is easy to see that the possibility of vertical portions in the form of intervals is excluded, since otherwise $\bar{u}$ becomes independent of z in ω_- over such portions, contradicting strict monotonicity of $\bar{u}$ there.

Note that in the case $\Sigma = \mathbb{R}$ we recover the classical result of combustion theory [16, 64]

$$\bar{u}(z) = \begin{cases} e^{-\sqrt{2}z}, & z > 0, \\ 1, & z \leq 0. \end{cases} \tag{5.17}$$

which is the minimizer with speed $c^\dagger = \sqrt{2}$. We note that by the same arguments as in Proposition 4.4, this is also the minimizer in the case of purely Neumann boundary conditions (i.e. $\partial\Sigma_\pm = \varnothing$) and is the fastest variational traveling wave irrespectively of the choices of φ, Ω, and the boundary conditions (see below).

Throughout the rest of this section we always assume that $n \geq 2$.

Remark 5.4. *Using simple test functions, one can show that condition in (5.16) of Corollary 5.3 is satisfies whenever Ω contains a ball of radius R big enough.*

Let us now derive an area-type functional which can be used to obtain suitable bounds for the propagation speed of the minimizer. Integrating the first term in (5.8) by parts and the second term of (5.8) in z, and using (5.5) and (5.13), we obtain

$$\Phi^0_{c^\dagger}[\bar{u}] = \int_{\partial\omega_\pm} e^{c^\dagger z + \varphi(y)} \left(\frac{|\nabla \bar{u}|}{2} - \frac{\nu \cdot \hat{z}}{c^\dagger} \right) d\mathcal{H}^{n-1} = 0, \tag{5.18}$$

where the gradient is evaluated on the ω_- side of $\partial\omega_\pm$. Then, making use of the free boundary conditions (5.13), we find

$$\Pi_{c^\dagger}(\partial\omega_\pm) = \Phi^0_{c^\dagger}[\bar{u}] = 0, \tag{5.19}$$

where we introduced an area-type functional

$$\Pi_c(\partial\omega_\pm) = \int_{\partial\omega_\pm} e^{cz+\varphi(y)} \left(\frac{1}{\sqrt{2}} - \frac{\nu \cdot \hat{z}}{c} \right) d\mathcal{H}^{n-1}, \tag{5.20}$$

where $\hat{z}$ is the unit vector along the z-axis pointing to the right. It follows that, if the functional Φ^0_c has a minimizer, then $\inf \Pi_c \leq 0$ for all $0 < c \leq c^\dagger$. Therefore, if one can show that for some c we have $\Pi_c > 0$ for every surface contained in $\omega_0 \times \mathbb{R}$, then this automatically implies that $c^\dagger < c$.

Notice that the first term in (5.20) is an area term, whereas the second is a volume term, which is of lower order with respect to the first one. In particular, from the regularity theory for minimal surfaces (see [1, 31]), it follows that any minimizer of Π_c is smooth out of a closed singular set of zero $\mathcal{H}^{n-1}$-measure.

Before undertaking a more detailed analysis, let us make several general remarks regarding the functional Π_c. First, it is clear from (5.20) that $c^\dagger \leq \sqrt{2}$ independently of φ. Indeed, in (5.20) $\nu \cdot \hat{z} \leq 1$ so the integrand is strictly positive for all $c > \sqrt{2}$. On the other hand, the upper bound $c = \sqrt{2}$ is achieved only if the front is planar, hence, only when $\partial\Sigma_\pm = \varnothing$. In this case $\bar{u}$ depends only on the z-variable and is given explicitly by (5.17), a version of the result of Proposition 4.4 for Φ^0_c.

We now proceed with the analysis of (5.20). For $\zeta \in BV(\omega_0)$, we define

$$\Xi_c[\zeta] = \int_{\omega_0} e^{\varphi(y)} \left(\sqrt{\frac{c^2\zeta^2 + |\nabla_y \zeta|^2}{2}} - \zeta \right) dy. \tag{5.21}$$

Notice that there is a standard way to define the functional (5.21) on the whole of $BV(\omega_0)$ (see [3, Section 5.5]), as the lower semicontinuous relaxation of the same functional restricted to $H^1(\Omega)$, with Dirichlet boundary conditions on $\partial\omega_0 \setminus \partial\Omega_0$. In particular, the functional Ξ_c takes into account possible jumps of ζ inside ω_0 and on $\partial\omega_0 \setminus \partial\Omega_0$.

A simple calculation shows that, if $\zeta > 0$, we have

$$\Pi_c(\Gamma_{\frac{1}{c}\log\zeta}) = \frac{1}{c}\Xi_c[\zeta], \tag{5.22}$$

where Γ_h denotes the graph $z = h(y)$ for any $h \in BV(\omega_0)$. In fact, there is a one-to-one correspondence between the functions on which Ξ_c is defined and the hypersurfaces in the domain of definition of Π_c. Therefore, in the following we will be using these two area-type functionals interchangeably.

Notice that Ξ_c is a one-homogeneous, convex, lower semicontinuous functional on $BV(\omega_0)$. Moreover, its gradient term corresponds to an anisotropic perimeter of the subgraph of ζ. Reasoning as in [31], it is possible to prove that $\bar\zeta$ is (locally) of class $C^{2,\alpha}$ in the open set where $\bar\zeta > 0$. We observe that any minimizer $\bar\zeta$ may be discontinuous (and jump to zero) on the boundary of such set.

The above arguments apply when the minimizer $\bar\zeta$ exists, this, of course, may not happen for all $c > 0$. In fact, the following statement holds.

Proposition 5.5. *Assume that* (5.16) *holds. Then, there exists a unique value of $c = c^\sharp$, for which Ξ_c admits a non-trivial minimizer $\bar\zeta \in BV(\omega_0)$, with $\bar\zeta \geq 0$ in ω_0. Furthermore, $\Xi_{c^\sharp}[\bar\zeta] = 0$.*

Proof. To construct a minimizer of Ξ_c, we consider the following constrained variational problem

$$\text{minimize } \Xi_c[\zeta] \qquad \text{subject to} \qquad \zeta \geq 0\,, \quad \int_{\omega_0} e^{\varphi(y)}\zeta\, dy = 1. \tag{5.23}$$

Indeed, letting ζ_n be a minimizing sequence, we have $\|\zeta_n\|_{BV(\omega_0)} \leq C$ for some $C > 0$, hence there exists a function ζ_c such that, up to a subsequence, $\zeta_n \rightharpoonup \zeta_c$ weakly in $BV(\omega_0)$. In particular $\zeta_n \to \zeta_c$ strongly in $L^1(\omega_0)$, and so ζ_c satisfies the constraints. Since Ξ_c is a lower-semicontinuous functional on $BV(\omega_0)$ [3], we get that ζ_c is a minimizer for the problem.

For shorthand we set $\mu_c = \Xi_c[\zeta_c]$. Theorem 5.1, (5.19) and (5.22) imply that

$$\inf\left\{\Xi_{c^\dagger}[\zeta] :\ \zeta \in BV(\omega_0),\ \zeta \geq 0\right\} \leq 0, \tag{5.24}$$

hence $\mu_{c^\dagger} \leq 0$. Moreover, from the discussion preceding (5.22), we have $\mu_c > 0$ for all $c > \sqrt{2}$. Furthermore, μ_c is an increasing function of c, hence $\mu_c < 0$, for all $c \in [0, c^\dagger)$. Indeed, $\Xi_{c'}[\zeta_c] < \Xi_c[\zeta_c]$ for any $0 \leq c' < c$, due to the monotonicity of the integrand in (5.21). Also, since $\zeta_{c'}$ is a minimizer of $\Xi_{c'}$, we have

$$\mu_{c'} = \Xi_{c'}[\zeta_{c'}] \leq \Xi_{c'}[\zeta_c] < \Xi_c[\zeta_c] = \mu_c. \tag{5.25}$$

Furthermore, by Mean Value Theorem applied pointwise to the integrand, with some $\tilde c(y) \in (c', c)$, we obtain

$$\mu_{c'} - \mu_c \geq -\frac{c - c'}{\sqrt{2}} \int_{\omega_0} e^{\varphi(y)} \frac{\tilde c \zeta_{c'}^2}{\sqrt{\tilde c^2 \zeta_{c'}^2 + |\nabla_y \zeta_{c'}|^2}} dy \geq -\frac{c - c'}{\sqrt{2}}. \tag{5.26}$$

So, $c \mapsto \mu_c$ is continuous, and hence there exists a unique value of $c = c^\sharp$ such that $\mu_{c^\sharp} = 0$.

We now claim that $\bar{\zeta} = \zeta_{c^\sharp}$ is a minimizer of $\Xi_{c^\sharp}$ in the absence of the constraint. This follows from the fact that, for all $\zeta \in BV(\omega_0), \zeta \geq 0, \zeta \not\equiv 0$, we have

$$\Xi_c[\zeta] = a\Xi_c[\zeta/a], \qquad a = \int_{\omega_0} e^{\varphi(y)}\zeta \, dy > 0. \tag{5.27}$$

Hence, $\Xi_{c^\sharp} \geq 0$, and $\bar{\zeta}$ is a global minimizer of $\Xi_{c^\sharp}$. Moreover, if $c < c^\sharp$, then by (5.27) $\inf \Xi_c \leq a\mu_c \to -\infty$ as $a \to \infty$, and so the minimizer of Ξ_c does not exist. If, on the other hand, $c > c^\sharp$, then $\Xi_c[\zeta] = a\Xi_c[\zeta/a] \geq a\mu_c$, so that the only minimizer is the trivial one. □

We note that in general the support of $\bar{\zeta}$ (or even ω_0) does not have to be connected. However, on all connected portions of $\mathrm{supp}(\bar{\zeta})$ the functional $\Xi_{c^\sharp}$ must evaluate to zero, since otherwise it can be decreased by setting $\bar{\zeta}$ to zero in the one where it is positive. But this means that one can always choose a minimizer $\bar{\zeta}$ whose support is connected.

Let us now summarize the arguments leading from (5.20) and (5.22) to Proposition 5.5 in the following result:

Proposition 5.6. *Under the assumptions of Theorem 5.1, we have*

$$c^\dagger \leq c^\sharp, \tag{5.28}$$

where $c^\sharp$ is defined in Proposition 5.5.

Note that in the absence of information about the minimizers of E_0 it is still possible to use the functional Ξ_c to obtain a sufficient condition for non-existence of minimizers for Φ_c^0. Allowing the domain of the functions ζ to be the whole of Ω, we obtain that the condition

$$\inf_{\substack{\zeta \in BV(\Omega) \\ \zeta \geq 0}} \Xi_0[\zeta] = 0, \tag{5.29}$$

guarantees non-existence of minimizers for Φ_c^0 with any $c > 0$ in view of the monotonicity of Ξ_c with respect to Ω.

Let us point out that the minimizers of Ξ_c or, equivalently, of Π_c satisfy the Euler-Lagrange equation which reduces to the classical Markstein model of the dynamics of flame fronts [41]. This fact, for a thin flame in a potential flow was first noticed by Joulin [33], who introduced a functional which is essentially equivalent to Π_c. To see this, let us compute the first variation of $\Pi_c(\Gamma)$ with respect to infinitesimal displacements $\delta\rho$ of Γ along the unit normal vector ν pointing to the right. After simple manipulations, we arrive at

$$\delta\Pi_c(\Gamma) = \frac{1}{\sqrt{2}} \int_\Gamma e^{cz+\varphi(y)} \left(c\nu \cdot \hat{z} + \nu \cdot \nabla_y \varphi + (n-1)H - \sqrt{2} \right) \delta\rho \, d\mathcal{H}^{n-1}, \tag{5.30}$$

where H is the mean curvature of Γ, positive if Γ is convex towards $z = -\infty$. Therefore, if Γ is a minimizer of $\Pi_{c^\sharp}$, it satisfies the Euler-Lagrange equation

$$\nu \cdot (c^\sharp \hat{z} + \nabla_y \varphi) = \sqrt{2} - (n-1)H. \tag{5.31}$$

This is precisely the steady version of the Markstein equation, once it is realized that the term on the left is the normal velocity of the front relative to the flow. So, what we proved in Proposition 5.6 gives a rigorous justification for the Markstein model as giving a strict upper bound for the propagation speed of the flame front in the considered setup. On the other hand, the physical assumptions behind the Markstein model rely on the smallness of the front curvature and the flow variation compared to the width of the preheat zone [41]. Under this assumption, we can show that the minimizers of Π_c or, equivalently, of Ξ_c also give a matching *lower* bound for $c^\dagger$ which coincides with $c^\sharp$ in the limit of vanishing front curvature and advection velocity gradient. For clarity, we demonstrate this point under a simplifying assumption on the geometry of Ω.

Proposition 5.7. *Assume Theorem 5.1 holds, and, in addition, that* $\bar{\Gamma} = \Gamma_{\frac{1}{c}\log\bar{\zeta}}$*, where* $\bar{\zeta}$ *is a minimizer of* Ξ_c *obtained in Proposition 5.5, has all principal curvatures bounded by* $\varepsilon > 0$*, that* $\operatorname{dist}(\omega_0, \partial\Omega) = O(\varepsilon^{-1})$*, and that* $|(\nabla_y \otimes \nabla_y)\varphi| \leq M\varepsilon$ *for some* $M > 0$*. Then*

$$c^\dagger \geq c^\sharp - C_1\varepsilon^2 - C_2 M\varepsilon, \tag{5.32}$$

for some $C_{1,2} > 0$ *independent of* ε*, when* ε *is small enough.*

Proof. We prove this statement by constructing an appropriate trial function for Φ_c^0 from $\bar{\Gamma}$, based on the one-dimensional minimizer, see (5.17). Introduce the signed distance function

$$d(x) = \pm \operatorname{dist}(x, \bar{\Gamma}), \tag{5.33}$$

which is positive if x is to the right of $\bar{\Gamma}$ and negative otherwise. We can then define a trial function

$$u(x) = \begin{cases} 1, & d(x) \leq 0, \\ e^{-\sqrt{2} d(x)}, & 0 < d(x) < d_0 - 1, \\ e^{-\sqrt{2}(d_0 - 1)}(d_0 - d(x)), & d_0 - 1 \leq d(x) < d_0, \\ 0, & d(x) \geq d_0, \end{cases} \tag{5.34}$$

where we introduced a constant d_0 such that $1 < d_0 < \operatorname{dist}(\omega_0, \partial\Omega)$. Clearly, u lies in $H_c^1(\Sigma)$ and satisfies the boundary conditions on $\partial\Sigma$.

Let us now evaluate $\Phi_c^0[u]$ for some $c < c^\sharp$. To proceed, observe that the second term in (5.2) coincides with the second term in (5.20):

$$\int_\Sigma e^{cz + \varphi(y)} V_0(u)\, dx = -\frac{1}{c} \int_{\bar{\Gamma}} e^{cz + \varphi(y)} \nu \cdot \hat{z}\, d\mathcal{H}^{n-1}, \tag{5.35}$$

where, as before, ν is the unit normal vector to $\bar{\Gamma}$ pointing to the right. So, it remains to evaluate the first integral in (5.2). Let us write this integral in curvilinear coordinates associated with $\bar{\Gamma}$, which is justified when $d_0 \leq \varepsilon^{-1}$:

$$\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx = \int_{\bar{\Gamma}} \int_0^{d_0} e^{cz+\varphi(y)}|\nabla u(x)|^2 \left(\prod_{i=1}^{n-1}(1+k_i(x')\rho)\right) d\rho\, d\mathcal{H}^{n-1}(x'). \quad (5.36)$$

Here k_i are the principal curvatures on $\bar{\Gamma}$, assumed to be positive if the set enclosed by $\bar{\Gamma}$ (i.e. the set on the left of $\bar{\Gamma}$) is convex, and x' is the projection of x on $\bar{\Gamma}$, so that $x = x' + \rho\nu(x')$. Now we estimate

$$\begin{aligned}
&\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx \\
&\quad \leq \int_{\bar{\Gamma}} \int_0^{d_0} e^{cz+\varphi(y)}|\nabla u(x)|^2 (1+(n-1)H(x')\rho + C\varepsilon^2\rho^2)\, d\rho\, d\mathcal{H}^{n-1}(x') \\
&\quad \leq \int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_0^{d_0} e^{c\rho\nu\cdot\hat{z}+\rho\nu\cdot\nabla\varphi(y')+\varepsilon CM\rho^2} \\
&\qquad\qquad \times|\nabla u(x)|^2 (1+(n-1)H(x')\rho + C\varepsilon^2\rho^2)\, d\rho\, d\mathcal{H}^{n-1}(x') \\
&\quad \leq \int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_0^{d_0} e^{\sqrt{2}\rho+\varepsilon CM\rho^2}|\nabla u(x)|^2 (1+C\varepsilon^2\rho^2)\, d\rho\, d\mathcal{H}^{n-1}(x'), \quad (5.37)
\end{aligned}$$

where $H(x')$ is the mean curvature of $\bar{\Gamma}$ at x', and C denotes a generic positive constant. In writing the last line in the estimate above we took into account the Euler-Lagrange equation (5.31) for $\bar{\Gamma}$ and the fact that $e^{(c-c^\sharp)\rho\nu\cdot\hat{z}} \leq 1$. Substituting the expression for u from (5.34) and choosing $d_0 = K\log\varepsilon^{-1}$, with $K > 0$ sufficiently large, we get

$$\begin{aligned}
&\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx \\
&\quad \leq 2\int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_0^{d_0-1} e^{-\sqrt{2}\rho+\varepsilon CM\rho^2}(1+C\varepsilon^2\rho^2)\, d\rho\, d\mathcal{H}^{n-1}(x') \\
&\quad + \int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_{d_0-1}^{d_0} e^{\sqrt{2}(2-d_0)+\varepsilon CMd_0^2}(1+C\varepsilon^2 d_0^2)\, d\rho\, d\mathcal{H}^{n-1}(x') \\
&\quad \leq C\varepsilon^2 \int_{\bar{\Gamma}} e^{cz'+\varphi(y')} d\mathcal{H}^{n-1}(x') \qquad (5.38) \\
&\quad +2\int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_0^{d_0-1} e^{-\sqrt{2}\rho+\varepsilon CM\rho^2}\, d\rho\, d\mathcal{H}^{n-1}(x') \\
&\quad \leq (C_1\varepsilon^2 + C_2M\varepsilon) \int_{\bar{\Gamma}} e^{cz'+\varphi(y')} d\mathcal{H}^{n-1}(x') \\
&\quad +2\int_{\bar{\Gamma}} e^{cz'+\varphi(y')} \int_0^{\infty} e^{-\sqrt{2}\rho}\, d\rho\, d\mathcal{H}^{n-1}(x').
\end{aligned}$$

Integrating the last term with respect to ρ, we finally obtain

$$\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx \leq \sqrt{2}(1 + C_1\varepsilon^2 + C_2 M\varepsilon) \int_{\bar{\Gamma}} e^{cz+\varphi(y)} d\mathcal{H}^{n-1}. \tag{5.39}$$

Now, observe that from (5.22), Proposition 5.5, and the monotonicity of Ξ_c with respect to c it follows that

$$\frac{1}{\sqrt{2}} \int_{\bar{\Gamma}} e^{cz+\varphi(y)} d\mathcal{H}^{n-1} \leq \frac{1}{c^\sharp} \int_{\bar{\Gamma}} e^{cz+\varphi(y)} \nu \cdot \hat{z}\, d\mathcal{H}^{n-1}, \tag{5.40}$$

for all $0 < c \leq c^\sharp$. Combining this with the estimate in (5.39), we have

$$\begin{aligned} \Phi_c^0[u] &= \frac{1}{2} \int_\Sigma e^{cz+\varphi(y)} |\nabla u|^2 dx - \frac{1}{c} \int_{\bar{\Gamma}} e^{cz+\varphi(y)} \nu \cdot \hat{z}\, d\mathcal{H}^{n-1} \\ &\leq \frac{c - c^\sharp + C_3\varepsilon^2 + C_4 M\varepsilon}{c\sqrt{2}} \int_{\bar{\Gamma}} e^{cz+\varphi(y)} d\mathcal{H}^{n-1} = 0, \end{aligned} \tag{5.41}$$

if $c = c^\sharp - C_3\varepsilon^2 - C_4 M\varepsilon$. Therefore, u satisfies the assumptions of Theorem 5.1, and so there exists a minimizer with speed $c^\dagger \geq c$. $\square$

Let us make a few remarks about the result in Proposition 5.7 before concluding this section. The main assumption of Proposition 5.7 was that of uniform smallness of the curvature of $\bar{\Gamma}$, which is at the heart of the idea of approximating the free boundary $\partial\omega_\pm$ of the minimizer of Φ_c^0 with that of Π_c and is, therefore, essential here. We note that the assumption $\operatorname{dist}(\omega_0, \partial\Omega) = O(\varepsilon^{-1})$ does not contradict the assumption on the curvature, and may even be replaced by the weaker assumption $\operatorname{dist}(\omega_0, \partial\Omega) \gg \log\varepsilon^{-1}$ (see also Sec. 8). On the other hand, as follows from evaluating (5.31) at a point where $\nu = \hat{z}$, if the curvature of $\bar{\Gamma}$ is uniformly $O(\varepsilon)$, then the speed $c^\sharp$ has an estimate $c^\sharp = \sqrt{2} + O(\varepsilon)$ as well. But since $c^\sharp - c^\dagger = O(\varepsilon^2) + O(M\varepsilon)$, the speed $c^\sharp$ of the minimizer of Π_c captures, as it should, the leading order curvature corrections to $c^\dagger$ and so should give a good approximation for the propagation velocity $c^\dagger$ in practice. On the other hand, if $\bar{\Gamma}$ is allowed to approach $\partial\Sigma_\pm$ (the cold boundaries), then the curvature will not be uniformly bounded near the boundary, and the propagation speed can have an $O(1)$ difference from $c = \sqrt{2}$, the speed of the planar front, or propagation failure may occur altogether.

We also note, that if $|\nabla_y\varphi| \ll 1$, we get into the situation of a weakly perturbed planar front. Assuming that $|\nabla_y\zeta| \ll \zeta$ and $\sqrt{2} - c \ll 1$, Taylor-expanding (5.21) in $|\nabla_y\zeta|$, and introducing $\psi = \sqrt{\zeta}$, we obtain (up to an additive constant)

$$\Xi_c[\psi^2] = \int_{\omega_0} e^{\varphi(y)} \left(|\nabla_y\psi|^2 - \frac{\sqrt{2} - c}{\sqrt{2}} \psi^2 \right) dy + \text{h.o.t.} \tag{5.42}$$

Thus, in this situation finding $c^\sharp$ amounts to computing the smallest eigenvalue of the Schrödinger-type operator in (5.42), which is much easier than studying the minimizers of (5.21).

6 Approximating problems

Now we will study the question of how well the minimizers constructed in Sec. 5 approximate the minimizers of Sec. 4 in the sharp reaction zone limit. Our main results in this section are the estimates for the wave velocity $c_\varepsilon^\dagger$ of the approximating problem in terms of the speed $c_0^\dagger$ of the limit free boundary problem and strong convergence of the (appropriately translated) minimizers of the approximating problem to a solution of the limit free boundary problem.

Following up on the discussion at the end of Sec. 2, we will say that the problem

$$\Delta \bar{u}_\varepsilon + c\frac{\partial \bar{u}_\varepsilon}{\partial z} + \nabla_y \varphi \cdot \nabla_y \bar{u}_\varepsilon + f_\varepsilon(\bar{u}_\varepsilon) = 0, \quad \bar{u}_\varepsilon\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla \bar{u}_\varepsilon\big|_{\partial\Sigma_0} = 0 \quad (6.1)$$

approximates the sharp reaction zone limit considered in Sec. 5, if, for $0 < \varepsilon \ll 1$, the nonlinearity f_ε can be represented as

$$f_\varepsilon(u) = \varepsilon^{-1} g\left(\frac{1-u}{\varepsilon}\right), \quad (6.2)$$

where the function $g \in C^{1,\gamma}(\mathbb{R})$ has the following properties:

$$g \geq 0, \qquad \text{supp } g \subseteq [0,1], \qquad \int_0^1 g(u)du = 1. \quad (6.3)$$

It is not difficult to see that with these definitions we have $f_\varepsilon(u) \to \delta(u - 1^-)$, consistent with the discussion at the beginning of Sec. 5.

We note that this type of an approximation was analyzed by Berestycki, Caffarelli and Nirenberg in [7], where they used this problem as a regularising approximation to construct the solutions of the limiting free boundary problem and get information on the regularity of the free boundary, for Neumann boundary condition on $\partial\Sigma$ and in the presence of a shear flow along z. We, on the other hand, will take a different approach, in view of our a priori existence results for the free boundary problem, and will show how the latter can be used to approximate the solutions of (6.1). Also, our analysis treats a more general class of boundary conditions and a transverse potential flow, instead of a shear axial flow.

Let us define

$$V_\varepsilon(u) = -\int_0^u f_\varepsilon(s)\, ds. \quad (6.4)$$

Then it immediately follows that $V_\varepsilon(0) = dV_\varepsilon(0)/du = 0$ and

$$V_\varepsilon(u) \leq V_0(u), \qquad \lim_{\varepsilon\to 0} V_\varepsilon(u) = V_0(u), \qquad \forall u \in \mathbb{R}, \quad (6.5)$$

where V_0 is defined in (5.1). So, if we define the functional

$$\Phi_c^\varepsilon[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla u|^2 + V_\varepsilon(u)\right) dx, \quad (6.6)$$

then by the results of Sec. 4 existence of minimizers for Φ_c^ε implies existence of solutions $\bar{u}_\varepsilon \in H_c^1(\Sigma)$ for (6.1). Moreover, under the assumption of existence of minimizers for the limit functional Φ_c^0 in (5.2) existence of minimizers for Φ_c^ε is guaranteed for all $\varepsilon < 1$. Indeed, by Corollary 5.3, we have $\inf E_0[v] < 0$, and, from the first inequality in (6.5), that $\inf E_\varepsilon[v] < 0$ as well, where E_ε is given by (3.10) with V replaced by V_ε. Since, by (6.2) and (6.3) we have $df_\varepsilon(0)/du = 0$ for $\varepsilon < 1$, this implies that $\nu_0^\varepsilon \geq 0$, where, once again ν_0^ε is the same as ν_0 in (3.9), if f is replaced by f_ε in (4.5). So, by Theorem 4.9 the minimizer $\bar{u}_\varepsilon$ of Φ_c^ε exists and has all the properties guaranteed by Theorem 4.3.

We now show that the speed $c_0^\dagger$ is in fact the limiting speed of the minimizers $\bar{u}_\varepsilon$ for the approximating problem.

Proposition 6.1. *Under the assumption of Corollary 5.3, let $c_0^\dagger$ and $c_\varepsilon^\dagger$ be the speeds of the minimizers of Φ_c^0 and Φ_c^ε, respectively. Then we have*

$$c_0^\dagger \leq c_\varepsilon^\dagger \leq c_0^\dagger + \frac{32\varepsilon}{c_0^\dagger}, \qquad 0 < \varepsilon < \frac{1}{2}. \tag{6.7}$$

Proof. Since $V_\varepsilon \leq V_0$, we immediately conclude the lower bound in (6.7). Let us now prove the upper bound. It is easy to see that by the assumptions on f_ε we have $V_\varepsilon(u) \geq V_0\left(\frac{u}{1-\varepsilon}\right)$. Let us introduce $\tilde{u}(y,z) = \frac{1}{1-\varepsilon} u\left(y, \frac{c_0^\dagger}{c} z\right)$. Then, clearly for any $u \in H_c^1(\Sigma)$ we have $\tilde{u} \in H^1_{c_0^\dagger}(\Sigma)$. So, evaluating Φ_c^ε on u, we get

$$\Phi_c^\varepsilon[u] \geq \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{u_z^2}{2} + \frac{|\nabla_y u|^2}{2} + V_0\left(\frac{u}{1-\varepsilon}\right) \right\} dx$$

$$\geq \left(\frac{c_0^\dagger}{c}\right) \int_\Sigma e^{c_0^\dagger z+\varphi(y)} \left\{ (1-\varepsilon)^2 \frac{|\nabla_y \tilde{u}|^2}{2} + (1-\varepsilon)^2 \left(\frac{c}{c_0^\dagger}\right)^2 \frac{u_z^2}{2} + V_0(\tilde{u}) \right\} dx$$

$$\geq (1-\varepsilon)^2 \left(\frac{c_0^\dagger}{c}\right) \left(\Phi^0_{c_0^\dagger}[\tilde{u}] \right.$$

$$\left. + \int_\Sigma e^{c_0^\dagger z+\varphi(y)} \left\{ \frac{c^2 - {c_0^\dagger}^2}{2{c_0^\dagger}^2} \tilde{u}_z^2 + \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2} V_0(\tilde{u}) \right\} dx \right). \tag{6.8}$$

Now, using the fact that $V_0(u) \geq -u^2$, the Poincaré inequality from (3.6), and that by definition of $c_0^\dagger$ we have $\Phi^0_{c_0^\dagger}[\tilde{u}] \geq 0$ for all $\tilde{u} \in H^1_{c_0^\dagger}(\Sigma)$, we can proceed to estimate the last line in the inequality above as

$$\Phi_c^\varepsilon[u] \geq (1-\varepsilon)^2 \left(\frac{c_0^\dagger}{c}\right) \int_\Sigma e^{c_0^\dagger z+\varphi(y)} \left(\frac{c^2 - {c_0^\dagger}^2}{8} - \frac{2\varepsilon}{(1-\varepsilon)^2} \right) \tilde{u}^2 \, dx. \tag{6.9}$$

Then, from this inequality it follows that $\Phi_c^\varepsilon[u] \geq \delta \int_\Sigma e^{cz+\varphi(y)} u^2 \, dx$, with some $\delta > 0$, if $\varepsilon < \frac{1}{2}$ and $c > c_0^\dagger + \frac{32\varepsilon}{c_0^\dagger}$, so only trivial minimizers exist for these values of c. In view of this, we have the second inequality in (6.7).

□

Let us recall the following uniform gradient estimate for both the minimizers $\bar{u}_\varepsilon$ of Φ_c^ε and the minimizer $\bar{u}_0$ of the limit functional Φ_c^0, which were obtained in [7, Theorem 3.1] (see also [17, Chapter 1]).

Proposition 6.2. *There exists a constant $C > 0$ independent of ε such that*

$$\|\bar{u}_\varepsilon\|_{W^{1,\infty}(\Sigma)} \leq C, \qquad \|\bar{u}_0\|_{W^{1,\infty}(\Sigma)} \leq C. \tag{6.10}$$

With the help of these estimates, we are now ready to prove our convergence result for the sequence of minimizers Φ_c^ε of the approximating problem to a minimizer of Φ_c^0.

Proposition 6.3. *There exists a translation $a_\varepsilon \in \mathbb{R}$, such that if $u_\varepsilon(y,z) = \bar{u}_\varepsilon(y, z - a_\varepsilon)$, then we have*

$$u_\varepsilon \to u_0 \in C^{0,1}(\overline{\Sigma}) \quad \text{in} \quad H^1_{c_0^\dagger}(\Sigma), \tag{6.11}$$

as $\varepsilon \to 0$, and

$$\Phi^0_{c_0^\dagger}[u_0] = 0, \qquad u_0 \not\equiv 0. \tag{6.12}$$

The convergence is also uniform on compact subsets of $\overline{\Sigma}$.

Proof. Let $\varepsilon < \frac{1}{2}$, and observe that we have $\sup_{x\in\Sigma} \bar{u}_\varepsilon(x) > \frac{1}{2}$, since otherwise $V_\varepsilon(\bar{u}_\varepsilon) \equiv 0$ and so $\Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] > 0$, which contradicts the fact that u_ε is a minimizer of $\Phi^\varepsilon_{c_\varepsilon^\dagger}$. Recalling also that $\bar{u}_\varepsilon(\cdot, z) \to 0$ uniformly as $z \to +\infty$, we can choose $a_\varepsilon \in \mathbb{R}$ such that, by letting $u_\varepsilon(y,z) = \bar{u}_\varepsilon(y, z - a_\varepsilon)$, we have

$$\max_{y\in\overline{\Omega}} u_\varepsilon(y,0) = \frac{1}{2} \qquad \text{and} \qquad u_\varepsilon(y,z) \leq \frac{1}{2}, \quad \forall (y,z) \in \Omega \times [0,+\infty). \tag{6.13}$$

Now, notice that by Proposition 6.2 and the Arzelà-Ascoli Theorem the functions u_ε converge (on a sequence of $\varepsilon \to 0$) uniformly on compact subsets of $\overline{\Sigma}$ to a function $u_0 \in C^{0,1}(\overline{\Sigma})$, which satisfies (6.13) (hence, in particular, $u_0 \not\equiv 0$). Moreover, from Lemma 3.3 and Proposition 6.1 we know that $c_\varepsilon^\dagger \geq c_0^\dagger$ and

$$u_\varepsilon \in H^1_{c_0^\dagger}(\Sigma), \qquad \forall \varepsilon > 0. \tag{6.14}$$

Let us show that u_ε are also equibounded in $H^1_{c_\varepsilon^\dagger}(\Sigma)$, and, as a consequence, in $H^1_{c_0^\dagger}(\Sigma)$ as well. Thanks to Lemma 3.2, it is enough to prove that

$$\int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx \leq C, \tag{6.15}$$

for some constant $C > 0$ independent of ε. Since $V_\varepsilon(u) \geq -1$ for all u and, by construction, $V_\varepsilon(u_\varepsilon(\cdot, z)) = 0$ for all $z > 0$, we have

$$\begin{aligned} 0 = \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] &= \frac{1}{2} \int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx + \int_{-\infty}^0 \int_\Omega e^{c_\varepsilon^\dagger z + \varphi(y)} V_\varepsilon(u_\varepsilon) \, dy dz \\ &\geq \frac{1}{2} \int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx - \frac{M|\Omega|}{c_\varepsilon^\dagger}, \end{aligned} \tag{6.16}$$

for some $M > 0$, which proves the inequality in (6.15) with $C = 2M|\Omega|/c_0^\dagger$. Now, to pass to the norm in $H^1_{c_0^\dagger}(\Sigma)$, we observe that

$$\int_\Sigma e^{c_0^\dagger z+\varphi(y)}|\nabla u_\varepsilon|^2 dx \leq \int_0^\infty \int_\Omega e^{c_\varepsilon^\dagger z+\varphi(y)}|\nabla u_\varepsilon|^2 dydz$$

$$+\int_{-\infty}^0 \int_\Omega e^{c_0^\dagger z+\varphi(y)}|\nabla u_\varepsilon|^2 dydz \leq C + \frac{M|\Omega|}{c_0^\dagger}||\,|\nabla u_\varepsilon|^2||_{L^\infty(\Sigma)}. \tag{6.17}$$

In view of the result in Proposition 6.2, we conclude that u_ε are equibounded in $H^1_{c_0^\dagger}(\Sigma)$ also. So, it follows that $u_\varepsilon \rightharpoonup u_0$ also weakly in $H^1_{c_0^\dagger}(\Sigma)$.

Let us now prove that

$$\Phi^0_{c_0^\dagger}[u_0] = 0. \tag{6.18}$$

Since we already know that $\Phi^0_{c_0^\dagger}[u] \geq 0$ for all $u \in H^1_{c_0^\dagger}(\Sigma)$, in order to obtain (6.18) it is enough to prove that

$$\Phi^0_{c_0^\dagger}[u_0] \leq \lim_{\varepsilon\to 0} \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] = 0. \tag{6.19}$$

Recalling (6.13), for $\varepsilon < \frac{1}{2}$ we can write

$$\begin{aligned} 0 = \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] \quad \geq \quad & \int_{-\infty}^0 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{(c_\varepsilon^\dagger - c_0^\dagger)z} e^{c_0^\dagger z+\varphi(y)}\, dydz \\ & + \int_0^{+\infty} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{c_0^\dagger z+\varphi(y)}\, dydz \\ & + \int_{-\infty}^0 \int_\Omega V_\varepsilon(u_\varepsilon) e^{c_\varepsilon^\dagger z+\varphi(y)}\, dydz. \end{aligned} \tag{6.20}$$

Now, when $u_0 < 1$, by definition of V_ε we have $V_\varepsilon(u_\varepsilon) \to 0 = V_0(u_0)$. Since also $V_\varepsilon(u_\varepsilon) \geq -1 = V_0(u_0)$ whenever $u_0 = 1$, this implies that $V_0(u_0) \leq \liminf_{\varepsilon\to 0} V_\varepsilon(u_\varepsilon) \leq 0$ in Σ. Then, in view of $e^{c_\varepsilon^\dagger z} \to e^{c_0^\dagger z}$ by Proposition 6.1, it follows that

$$e^{c_0^\dagger z+\varphi(y)} V_0(u_0) \leq \liminf_{\varepsilon\to 0} e^{c_\varepsilon^\dagger z+\varphi(y)} V_\varepsilon(u_\varepsilon) \quad \text{ in } \Omega\times(-\infty,0). \tag{6.21}$$

Notice also that $e^{c_\varepsilon^\dagger z+\varphi(y)} V_\varepsilon(u_\varepsilon) \geq -e^{c_0^\dagger z+\varphi(y)} \in L^1(\Omega\times(-\infty,0))$. By monotonicity of u_0, we have $V_0(u_0) = V_\varepsilon(u_\varepsilon) = 0$ in $\Omega\times(0,+\infty)$. So, by Fatou's Lemma we finally obtain

$$\int_\Sigma V_0(u_0) e^{c_0^\dagger z+\varphi(y)}\, dx \leq \liminf_{\varepsilon\to 0} \int_\Sigma V_\varepsilon(u_\varepsilon) e^{c_\varepsilon^\dagger z+\varphi(y)}\, dx. \tag{6.22}$$

Similarly, since $e^{\frac{c_\varepsilon^\dagger - c_0^\dagger}{2} z} \to 1$ in $L^2_{c_0^\dagger}\big(\Omega\times(-\infty,0)\big)$, and

$$\nabla u_\varepsilon \quad \rightharpoonup \quad \nabla u_0 \quad \text{weakly in } L^2_{c_0^\dagger}\big(\Omega\times(0,+\infty);\mathbb{R}^n\big), \tag{6.23}$$

we have

$$\nabla u_\varepsilon e^{\frac{c_\varepsilon^\dagger - c_0^\dagger}{2} z} \quad \rightharpoonup \quad \nabla u_0 \quad \text{weakly in } L^2_{c_0^\dagger}(\Omega \times (-\infty, 0); \mathbb{R}^n), \tag{6.24}$$

which implies

$$\int_\Sigma \frac{|\nabla u_0|^2}{2} e^{c_0^\dagger z + \varphi(y)}\, dx \le \liminf_{\varepsilon \to 0} \int_0^{+\infty} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{c_0^\dagger z + \varphi(y)}\, dydz + \liminf_{\varepsilon \to 0} \int_{-\infty}^0 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{(c_\varepsilon^\dagger - c_0^\dagger) z} e^{c_0^\dagger z + \varphi(y)}\, dydz. \tag{6.25}$$

Taking the lim inf in (6.20) and recalling (6.22) and (6.25), we get the equality (6.18). Finally, in view of (6.13), we then obtain that u_0 is a nontrivial minimizer of Φ_c^0.

Notice that by (6.18) the inequalities in (6.22) and (6.25) are in fact equalities, therefore we also have $u_\varepsilon \to u_0$ strongly in $H^1_{c_0^\dagger}(\Sigma)$, as $\varepsilon \to 0$. Also, in view of the uniqueness of the minimizer of Φ_c^0 subject to (6.13) (recall that by Theorem 5.1 the minimizer u_0 is strictly decreasing whenever $u_0 < 1$), the limit is a full limit and is not restricted to a sequence of $\varepsilon \to 0$. $\square$

7 Propagation

We are now going to study the role the traveling waves constructed in the preceding sections play for the initial value problem governed by (1.1). Our main result in this section is that, under certain generic assumptions on the initial data, the solutions of the initial value problem propagate asymptotically with the speed $c^\dagger$ of the minimizers or, if these do not exist, with speed c_0, for sufficiently localized initial data.

In order to proceed, we first need to set up a suitable existence theory for the initial value problem associated with (1.1). This is relatively standard, except for the fact that we want also to have control on the behavior of solutions at $z = +\infty$ to ensure that the solutions stay in the spaces $H^1_c(\Sigma)$ with appropriate values of c. This is needed in order to be able to apply the energy methods associated with the functional Φ_c evaluated on the solutions of (1.1).

We start with the following basic result that guarantees existence of solutions for the initial value problem in (1.1) for initial data with sufficiently rapid exponential decay.

Proposition 7.1. *Let $c > 0$ and let $u_0 \in UC(\overline{\Sigma})$, where $UC(\overline{\Sigma})$ denotes the space of uniformly continuous functions on $\overline{\Sigma}$. Let also u_0 satisfy the boundary conditions in (1.4) and assume $u_0(x) \in [0,1]$ for all $x \in \Sigma$. Then there exists a unique solution $u \in C_1^2(\Sigma \times (0,\infty)) \cap C^0(\overline{\Sigma} \times [0,+\infty))$ of (1.1) with boundary conditions from (1.4), which satisfies $u(\cdot,0) = u_0$, $u(x,t) \in [0,1]$, for all $x \in \Sigma$ and $t > 0$, and $\|\nabla u\|_{L^\infty(\Sigma \times (\delta,+\infty))} < \infty$, for all $\delta > 0$. Moreover, if $u_0 \in L^2_c(\Sigma)$, we also have $u \in C^\alpha((0,+\infty); H^2_c(\Sigma)) \cap C^{1,\alpha}((0,+\infty); L^2_c(\Sigma))$, for all $\alpha \in (0,1)$,* *where $H^2_c(\Sigma)$ denotes the space of functions with up to second derivatives in $L^2_c(\Sigma)$.*

Proof. The result follows by standard theory of analytic semigroups (see e.g. [39]). The existence of a unique solution $u \in C^2_1(\Sigma\times(0,+\infty))\cap C^0(\overline{\Sigma}\times[0,+\infty))$ follows as in [39, Proposition 7.3.1], which can be extended to a cylindrical domain with boundary of class C^2. The estimate $u(\cdot,t) \in [0,1]$ follows from the Comparison Principle for parabolic equations [47, Chapter 3], since $u \equiv 0$ and $u \equiv 1$ are sub- and supersolution of (1.1), respectively. As a consequence we obtain that $\|\nabla u\|_{L^\infty(\Sigma\times(\delta,+\infty))} < \infty$, for all $\delta > 0$.

Let now $u_0 \in L^2_c(\Sigma)$, and denote by $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2_c(\Sigma)$ the linear operator $\mathcal{A}u = \Delta u + cu_z + \nabla_y\varphi \cdot \nabla_y u$, with $u \in \mathcal{D}(\mathcal{A}) = H^2_c(\Sigma) \subset L^2_c(\Sigma)$. Since $\mathcal{A}$ is a sectorial operator in $L^2_c(\Sigma)$, and $u \mapsto f(u,y)$ is (after an appropriate extension outside $[0,1]$) a Lipschitz map from $L^2_c(\Sigma)$ into itself, it follows from [39, Proposition 7.1.10] extended to a cylindrical domain that $u \in C^\alpha((0,+\infty); H^2_c(\Sigma)) \cap C^{1,\alpha}((0,+\infty); L^2_c(\Sigma))$, for all $\alpha \in (0,1)$. □

Proposition 7.2. *Let $u = u(y,z,t)$ be the solution of the initial value problem for (1.1), whose existence is guaranteed by Proposition 7.1. Then $\Phi_c[u(y,z+ct,t)]$ is a non-increasing function of t. In particular, if $\Phi_c[u(y,z,t_0)] < 0$ for some $t_0 \geq 0$, then $\Phi_c[u(y,z,t)] < 0$ for all $t > t_0$.*

Proof. We consider $\tilde{u}(y,z,t) = u(y,z+ct,t)$. This function satisfies the equation

$$\tilde{u}_t = \Delta\tilde{u} + c\tilde{u}_z + \nabla_y\varphi \cdot \nabla_y\tilde{u} + f(\tilde{u},y). \tag{7.1}$$

Since (7.1) is a gradient flow generated by Φ_c on $L^2_c(\Sigma)$ and $\tilde{u}_t(\cdot,t) \in L^2_c(\Sigma)$, we have

$$\frac{d\Phi_c[\tilde{u}(\cdot,t)]}{dt} = -\int_\Sigma e^{cz+\varphi(y)}\tilde{u}^2_t(\cdot,t)\,dx \leq 0, \tag{7.2}$$

and so $\Phi_c[\tilde{u}(\cdot,t)]$ is a non-increasing function of t. Since $\tilde{u}(y,z,t) = u(y,z+ct,t)$, the first statement follows immediately. Now, by monotonicity we find that $\Phi_c[\tilde{u}(y,z,t)] \leq \Phi_c[\tilde{u}(y,z,t_0)] = e^{-c^2t}\Phi_c[u(y,z,t_0)] < 0$. Therefore, we also have $\Phi_c[u(y,z,t)] = e^{c^2t}\Phi_c[\tilde{u}(y,z,t)] < 0$. □

Let us notice that this property of the sign of the functional Φ_c evaluated on the solution of (1.1) and its far-reaching consequences for propagation phenomena governed by (1.1) was first pointed out in [44]. We now show how this information can be used to obtain sharp upper and lower bounds for the propagation speed of the solution's *leading edge* in the considered problem. We will first consider the case in which Φ_c has minimizers $(c^\dagger,\bar{u})$, with $\lim_{z\to-\infty}\bar{u}(\cdot,z) = v$, as in Theorem 4.3. We define the position of the leading edge as in [37, 44]:

$$R_\delta(t) = \sup\{z \in \mathbb{R} : u(y,z,t) > \delta,\ \forall y \in \Omega\}, \tag{7.3}$$

for some small enough fixed $\delta > 0$, setting $R_\delta = -\infty$ if this set is empty.

Our first result establishes $c^\dagger$ as the upper bound for the speed of the leading edge for the initial data with sufficiently fast decay (see also [38, 44]).

Proposition 7.3. *Under the assumptions of Theorem 4.3, let u_0 satisfy the assumptions of Proposition 7.1 with some $c > c^\dagger$. Then, for any $\delta > 0$ we have $R_\delta(t) < c't$ for any $c' > c^\dagger$ and for all $t \geq T$, where $T = T(c') \geq 0$.*

Proof. First fix any $c' \in (c^\dagger, c)$ and $c'' \in (c^\dagger, c')$, then, according to Lemma 3.3, $u(\cdot, t) \in H^1_{c''}(\Sigma)$ as well, and Proposition 7.2 holds for $c = c''$. So, $\Phi_{c''}[u(y, z + c''t, t)]$ is a non-increasing function of t, and so $0 \leq \Phi_{c''}[u(y, z + c''t, t)] \leq C$ for all $t \geq t_0 > 0$. This, in turn, implies that $\Phi_{c''}[u(y, z + c't, t)] = e^{-c''(c'-c'')t}\Phi_{c''}[u(y, z + c't, t)] \to 0$. Arguing as in [38, Proposition 6.10], we conclude that $u(y, z + c't, t) \to 0$ in $L^2_{c''}(\Sigma)$, and in view of the uniform gradient estimate of Proposition 7.1 this means that $u(\cdot, t)$ converges to zero uniformly on the set $\overline{\Omega} \times [c't, +\infty)$ as $t \to \infty$. Therefore, there exists $T \geq 0$ such that $R_\delta(t) < c't$ for all $t > T$. Since this statement remains true also for all $c' > c$, this completes the proof. □

We point out that the proof of Proposition 7.3 relied only on the property $\Phi_c[u] \geq 0$ for all $u \in H^1_c(\Sigma)$ for all $c > c^\dagger$. On the other hand, if hypothesis (H3) is false, then the same is true for all $c > c_0$, with c_0 given by (4.51). This leads to the following extension of the result of Proposition 7.3 to the situation in which minimizers of Φ_c do not exist.

Corollary 7.4. *Under the assumptions of Theorem 4.11, let u_0 satisfy the assumptions of Proposition 7.1 with $c > c_0$. Then, for any $\delta > 0$ we have $R_\delta(t) < c't$ for any $c' > c_0$ and all $t \geq T$, where $T = T(c') \geq 0$.*

Let us also note that when the condition in (4.45) is not satisfied, we can use the same argument to show that $R_\delta(t) < c't$ for any $c' > 0$ and large enough t, consistent with the conclusion of [44] that propagation with finite speed is impossible in this situation.

Now we are going to study sufficient conditions for propagation. We point out right away that in general it is not clear whether a particular initial condition will result in a solution which propagates with non-zero velocity at long times. For example, if f is of bistable type, then propagation is clearly impossible for sufficiently small initial data, since they will rather decay to zero. In their classical work, Aronson and Weinberger gave a comprehensive study of propagation phenomena for scalar reaction-diffusion equations under various assumptions on the nonlinearity f [4, 5]. Their results depend quite delicately on the properties of the traveling wave solutions admitted by these equations and involve extensive applications of Maximum Principle. Recently, a general notion of *wave-like solutions* of (1.1) was introduced in [44] that identifies a large class of solutions of gradient reaction-diffusion systems which are propagating in a certain generalized sense (see Theorem 4.7 of [44]). Under some extra assumptions on the nonlinearity, propagation in this generalized sense implies propagation in the sense similar to the one used by Aronson and Weinberger [37, 44].

Generally, different modes of propagation can occur in the presence of multiple traveling wave solutions. Therefore, it is reasonable to ask what part of the initial condition determines the final propagation speed when propagation does occur. What we will show below is that for sufficiently rapidly decaying initial data the propagation speed can be controlled by the behavior of the initial data at $z = -\infty$ for front-like initial data. We will also prove that the propagation speed can be controlled by the limit to which the solutions of (1.1) converge on compact sets.

We first give the result under the assumption of existence of minimizers of Φ_c.

Proposition 7.5. *Under the assumptions of Theorem 4.3, let u_0 satisfy the assumptions of Proposition 7.1 with $c = c^\dagger$, and also assume that $\liminf_{z\to-\infty} u_0(\cdot, z) \geq v$ of Theorem 4.3 uniformly in Ω. Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have $R_\delta(t) > ct$ for any $c \in (0, c^\dagger)$ and all $t \geq T$, where $T = T(c) \geq 0$.*

Proof. The proof relies on the idea behind Theorem 4.11 of [44]. Suppose $\tilde{u}(x, 0)$ satisfies the assumptions of Proposition 7.1 with $c < c^\dagger$, $c^2 + 4\nu_0 > 0$, and $\Phi_c[\tilde{u}(\cdot, 0)] < 0$. Then by Proposition 7.2 we have $\Phi_c[\tilde{u}(\cdot, t)] < 0$ for all $t > 0$ as well. Arguing as in [44, Theorem 4.11(3)], we conclude that the leading edge $\tilde{R}_\delta(t)$ of $\tilde{u}$ obeys $\tilde{R}_\delta(t) > ct + \tilde{R}_0$ for all $t \geq 0$ and all $0 < \delta \leq \delta_0$, with some $\delta_0 > 0$ and $\tilde{R}_0 \in \mathbb{R}$.

Now, let us show that for any $c < c^\dagger$ there exists a function $u_c \in C^1(\overline{\Sigma})$ with compact support such that $\Phi_c[u_c] < 0$. Indeed, multiplying (4.1) by $e^{cz+\varphi(y)}\bar{u}_z$ and integrating over Σ, after a number of integrations by parts we obtain (see also [38, 44])

$$\Phi_c[\bar{u}] = \frac{c - c^\dagger}{c} \int_\Sigma e^{cz+\varphi(y)} \bar{u}_z^2 \, dx < 0. \tag{7.4}$$

Therefore, approximating $\bar{u}$ by a function from $C^1(\overline{\Sigma})$ with compact support and taking into account continuity of Φ_c in $H^1_c(\Sigma)$, we obtain the desired function u_c.

Observe also that since $\bar{u}(\cdot, z) \leq v$ for all $z \in \mathbb{R}$, we can also choose u_c such that $u_c(\cdot, z) < v$ strictly. Then, since u_c has compact support, we have in fact $u_c \leq v - \varepsilon$ with some $\varepsilon > 0$. Therefore, in view of the uniformity of $\liminf_{z\to-\infty} u_0(\cdot, z)$ in Ω, there exists $R \in \mathbb{R}$ such that $u_0(y, z) \geq u_c(y, z - R)$ for all $(y, z) \in \Sigma$. Choosing $\tilde{u}(y, z, 0) = u_c(y, z - R)$, we obtain a subsolution that propagates as $\tilde{R}_\delta(t) > ct$ for any $c < c^\dagger$ and $t \geq T(c)$ (here we dropped $\tilde{R}_0$ in view of arbitrary closeness of c to $c^\dagger$). Therefore, $R_\delta(t) \geq \tilde{R}_\delta(t) > ct$ as well. □

Remark 7.6. *Note that in general the dependence of $R_\delta(t)$ on δ cannot be removed because of the possibility of stacked waves moving with different speeds in the presence of multiple equilibria of E [50, 60].*

Now, if the minimizers do not exist for Φ_c, then, as expected, a similar result holds for u as in Proposition 7.5.

Proposition 7.7. *Under the assumptions of Theorem 4.11, let u_0 satisfy the assumptions of Proposition 7.1 with $c = c_0$ and let $\liminf_{z\to\infty} u_0(\cdot, z) \geq v$ of Theorem 4.11 uniformly in Ω. Then, there exists $\delta_0 > 0$ such that for any $c < c_0$ we have $R_\delta(t) > ct$, for all $\delta \in (0, \delta_0)$ and for all $t \geq T$, where $T = T(c) \geq 0$.*

Proof. We first use the same approximation as in Theorem 4.11 to show that for any $c \in (0, c_0)$ there exists a function $u_c \in C^1(\Sigma)$ with compact support such that $\Phi_c[u_c] < 0$. Indeed, if u_ε is a minimizer of the approximating functional $\Phi^\varepsilon_{c^\dagger_\varepsilon}$ from the proof of Theorem 4.11, then by (7.4) we have $\Phi^\varepsilon_c[u_\varepsilon] < 0$ for all $c < c^\dagger_\varepsilon$. In view of the fact that $c^\dagger_\varepsilon \to c_0$ from below, and in view of the continuity of Φ^ε_c in $H^1_c(\Sigma)$, for any $c \in (0, c_0)$ there exists $\varepsilon > 0$ and $u_c \in C^1(\Sigma)$ with compact support such that $\Phi^\varepsilon_c[u_c] < 0$ also. Then, we prove the claim above by observing that $f_\varepsilon \geq f$, and so $\Phi_c[u_c] \leq \Phi^\varepsilon_c[u_c]$.

Now, by an appropriate translation we can choose $u_c(y, z) = 0$ in $\Omega \times (0, +\infty)$. Consider now minimizers $\bar{u}_c$ of Φ_c which vanish for all $z \geq 0$. By standard theory [20], these minimizers exist and are classical solutions of (4.1) when $z < 0$ that vanish at $z = 0$. Furthermore, by redefining $V(u(y, z), y) = V(v(y), y)$ for all $u(y, z) \geq v(y)$, we may conclude that $\bar{u}_c$ are bounded from above by v. In fact, by Maximum Principle, $\bar{u}_c > 0$ in $\Omega \times (-\infty, 0)$, since $u = 0$ is not a minimizer, in view of the fact that u_c is in the considered class and $\Phi_c[u_c] < 0$. So, $\bar{u}_c(y, z - ct)$ is a subsolution of (1.1), and also $u_0(y, z) > \bar{u}_c(y, z - R)$ for some $R \in \mathbb{R}$.

Finally, arguing as in [38, Proposition 6.6], there exists a sequence $z_n \to -\infty$ on which $\bar{u}_c(y, z_n) \to v_c$, where v_c is a positive critical point of E bounded above by v. Since zero is an isolated critical point of E for $\nu_0 < 0$ by Remark 4.2, we conclude that $\sup_{x\in\Sigma} \bar{u}_c(x)$ is uniformly bounded away from zero, independently of c, which finally yields the statement of the Proposition. □

Summarizing all the results obtained above, we have the following

Theorem 7.8. *Assume hypotheses (H1) and (H2) are satisfied. Let u_0 satisfy the assumptions of Proposition 7.1 with some $c > c^*$, where $c^* = c^\dagger$ if hypothesis (H3) is true, or $c^* = c_0$ if hypothesis (H3) is false and $\nu_0 < 0$. In addition, assume that $\liminf_{z\to-\infty} u_0(\cdot, z) \geq v$ uniformly in Ω, where v is defined in Theorem 4.3 or Theorem 4.11. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ and any $\varepsilon > 0$ it holds*

$$(c^* - \varepsilon)t < R_\delta(t) < (c^* + \varepsilon)t, \tag{7.5}$$

for all $t \geq T$, where $T = T(\varepsilon) \geq 0$.

Thus, the speed c^* in Theorem 7.8 has a meaning of the propagation speed for the solutions of (4.1) with the initial data whose decay is governed by the L^2_c-norm with c sufficiently large. These are the data that we call "sufficiently localized"; in particular, initial data that equal zero identically for large enough z automatically fall in this class. Let us mention here that this assumption on the decay of the initial data is in fact crucial: as is well-known, one can construct solutions which propagate faster than c^* when $\nu_0 < 0$, if the initial data are allowed to decay slower [14, 40, 42, 44, 51, 53].

Finally, let us discuss the situation in which u_0 is not a front-like function, contrary to the lim inf assumption of Theorem 7.8. Alternatively, it is a generic property of the parabolic PDE in (1.1) that u approaches a limit, say v, on compact sets as $t \to \infty$ [5]. Then, we can once again use the function u_c constructed in the proof of Proposition 7.5 as a subsolution for large enough times for the solutions of (1.1). Therefore, we get the following analog of Theorem 7.8 in this case:

Corollary 7.9. *The statement of Theorem 7.8 remains true, if the the condition* $\liminf_{z\to-\infty} u_0(y,z) \geq v(y)$ *is replaced with* $\liminf_{t\to\infty} u(y,z,t) \geq v(y)$ *uniformly on compact subsets of* Σ*. If, in addition,* $u_0(y,-z) \in L^2_c(\Sigma)$ *the same statement holds for* $u(y,-z,t)$*.*

The last statement above implies that under these assumptions a *pair* of counter-propagating fronts will develop, moving with the same speed c^* in both directions $z \to \pm\infty$.

8 Numerical examples

In this section, we illustrate the applicability of our theory with a few numerical examples. We will concentrate on the results of Sec. 5 as, on one hand, the sharp reaction zone limit is important for combustion applications, and, on the other, because in this case both upper and lower bounds for propagation speed of the minimizers are available, and so it is possible to check how well they fit the propagation speed both for the limit problem and its regularizing approximations.

For the sake of clarity, we will consider the simplest possible, yet non-trivial situation, namely that of front propagation along a two-dimensional strip: $\Sigma = (0, 2L) \times \mathbb{R}$, where $L > 0$, with Dirichlet boundary conditions. We note that in the case of a bistable nonlinearity and $\varphi = 0$ existence of traveling waves on a strip with Dirichlet boundary conditions was first proved by Gardner in [28].

Let us start by considering the minimizers in the sharp reaction zone limit in the absence of a flow, $\varphi = 0$. Here we only need to consider the problem on half of the domain: $(0, L) \times \mathbb{R}$, due to the obvious symmetry of the solution with respect to the transformation $y \to 2L - y$. According to Corollary 5.3, the minimizers of Φ^0_c exist if and only if (5.16) holds. Here we have explicitly, according to Remark 5.2,

$$v = \begin{cases} \sqrt{2}\, y, & 0 \leq y \leq L_0, \\ 1, & L_0 \leq y \leq L, \end{cases} \tag{8.1}$$

where $L_0 = 1/\sqrt{2}$, and $E_0[v] < 0$ whenever

$$L > \sqrt{2}. \tag{8.2}$$

So, the minimizers of Φ_c^0 exist if and only if the value of L is greater than this critical value. Also note that for every L satisfying (8.2) the critical point of E_0 is unique and is given by (8.1). Therefore, a pair $(c^\dagger, \bar{u})$, where $\bar{u}$ is a minimizer of $\Phi_{c^\dagger}^0$ is, in fact, the only (up to translations) traveling wave solution in the sharp reaction zone limit. In particular, the speed of the wave is unique and is given by $c^\dagger > 0$.

To obtain a lower bound for the propagation speed $c^\dagger$, we introduce a trial function $u = u_{\lambda,\mu,l}$, where

$$u_{\lambda,\mu,l}(y,z) = \begin{cases} 1, & l \leq y \leq L,\ z \leq \frac{\mu}{\lambda}(y-L), \\ e^{-\lambda z + \mu(y-L)}, & l \leq y \leq L,\ z \geq \frac{\mu}{\lambda}(y-L), \\ \frac{y}{l}, & 0 \leq y \leq l,\ z \leq \frac{\mu}{\lambda}(l-L), \\ \frac{y}{l} e^{-\lambda z + \mu(l-L)}, & 0 \leq y \leq l,\ z \geq \frac{\mu}{\lambda}(l-L). \end{cases} \tag{8.3}$$

This function is characterized by 3 parameters, λ, μ, l. We must have $0 < l < L$, as well as $2\lambda > c$ in order for $u_{\lambda,\mu,l}$ to lie in $H_c^1((0,L) \times \mathbb{R})$. Substituting this u into Φ_c^0, after straightforward algebra we obtain

$$\begin{aligned} \Phi_c^0[u_{\lambda,\mu,l}] &= \frac{\left(-1 + e^{\frac{c(l-L)\mu}{\lambda}}\right)\left(\lambda^2 + \mu^2\right)\lambda}{2c(c-2\lambda)\mu} \\ &+ \frac{\left(-1 + e^{\frac{c(l-L)\mu}{\lambda}}\right)\lambda}{c^2\mu} - \frac{e^{\frac{c(l-L)\mu}{\lambda}}\left(l^2\lambda^2 + 3\right)}{6l(c-2\lambda)} + \frac{e^{\frac{c(l-L)\mu}{\lambda}}}{2cl}. \end{aligned} \tag{8.4}$$

With $c > 0$ and $L > \sqrt{2}$, this expression can be minimized numerically, and the sign of the minimum be evaluated. Then, one can find the largest value of c for which this minimum still remains reliably negative (to numerical precision). For example, when $L = \frac{5}{2}$ and $c = 0.925$, we found that $\Phi_c^0[u_{\lambda,\mu,l}]$ is minimized with $\lambda \simeq 1.237, \mu \simeq 0.5150, l \simeq 0.8870$ and attains the value of $\simeq -1.15 \times 10^{-3} < 0$. The level curves of $u_{\lambda,\mu,l}$ corresponding to these values are shown in Fig. 4(a). So, from Theorem 5.1 we conclude that $c^\dagger \geq 0.925$. Of course, it is no trouble at all to make this estimate completely rigorous, if need be.

Now we compute the upper bound $c^\sharp$ for the minimizer above. For that, we need to find a non-trivial minimizer ζ for the functional Ξ_c in (5.21), with $\omega_0 = (L_0, L)$, $\zeta_y(L) = 0$, and $\zeta(L_0) = 0$. In the case of a general potential $\varphi(y)$, the Euler-Lagrange equation for Ξ_c is

$$\frac{d}{dy}\left(\frac{e^{\varphi(y)}\zeta_y}{\sqrt{c^2\zeta^2 + \zeta_y^2}}\right) = e^{\varphi(y)}\left(\frac{c^2\zeta}{\sqrt{c^2\zeta^2 + \zeta_y^2}} - \sqrt{2}\right). \tag{8.5}$$

Actually, this equation can be solved in closed form in the special case when φ is a linear function of y and, in particular, when $\varphi = 0$ (of course, this equation can also be straightforwardly integrated numerically for arbitrary φ to any desired accuracy). However, since the algebra becomes too messy in the case $\varphi = \alpha y$

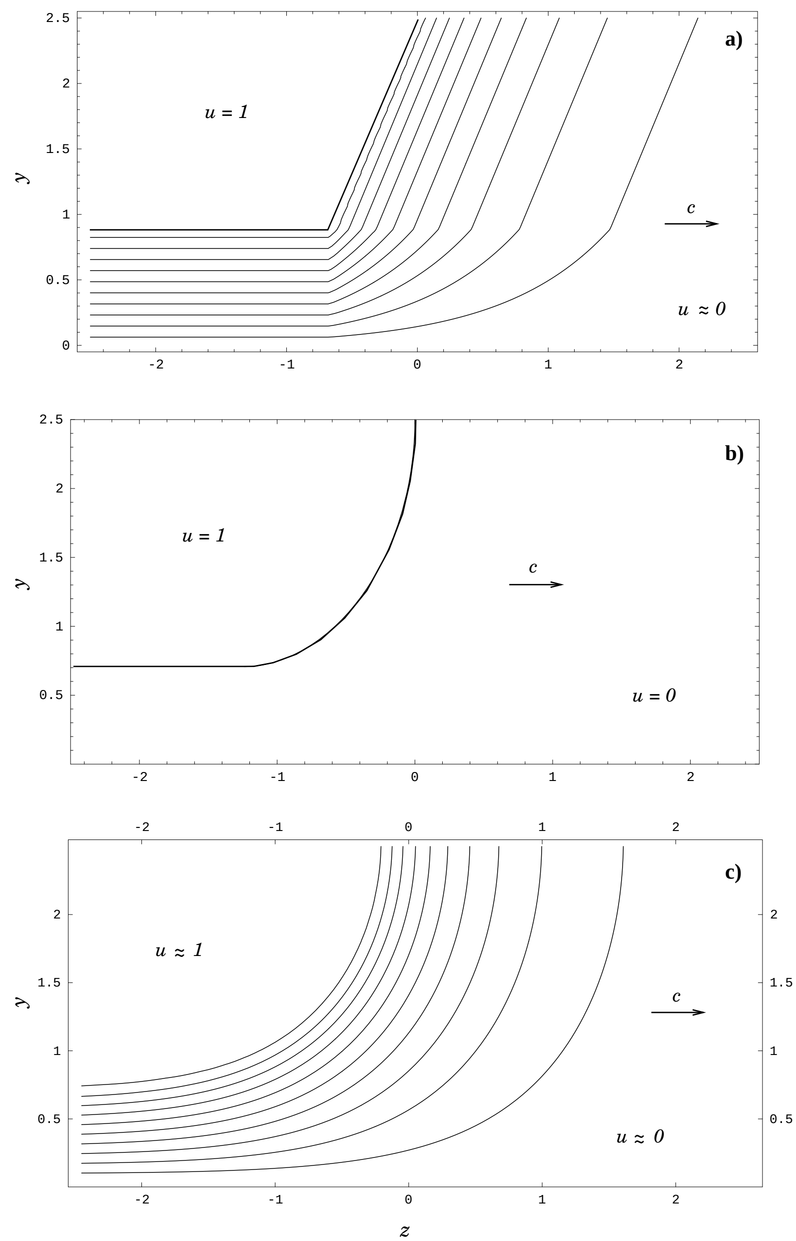

Figure 4: Comparison of the front profiles obtained from different approximations in the case $\varphi = 0$ and $L = \frac{5}{2}$. (a) The level curves of the trial function $u_{\lambda,\mu,l}$ for $c = 0.925$ and the parameters λ, μ, l obtained from minimizing $\Phi_c^0[u_{\lambda,\mu,l}]$. (b) The curve minimizing Π_c with $c = c^\sharp$. (c) The level curves of the numerical solution of (6.1) and (8.12) for $\varepsilon = 0.2$. Only the lower half of u is shown in all cases.

with $\alpha \neq 0$, we will only analyze the case $\varphi = 0$ explicitly, and will instead use a numerical solution of (8.5) in other cases.

When $\varphi = 0$, the first integral of (8.5) is

$$H = \sqrt{2}\,\zeta - \frac{c^2\zeta^2}{\sqrt{c^2\zeta^2 + \zeta_y^2}}. \tag{8.6}$$

Also, given H, the value of the functional on the solution of (8.6) is

$$\Xi_c[\zeta] = \frac{1}{\sqrt{2}} \left(\zeta(L_0 + 0) + \int_{L_0}^{L} \frac{\zeta_y^2\, dy}{\sqrt{c^2\zeta^2 + \zeta_y^2}} - H(L - L_0) \right), \tag{8.7}$$

where we took into account a jump discontinuity in ζ at $y = L_0$. Now, note that in view of (8.7), we will get $\Xi_c[\zeta] > 0$ unless $\zeta = 0$ when $H \leq 0$. Therefore, we need to consider only the case $H > 0$. In fact, because the right-hand side of (8.6) is a one-homogeneous function of ζ and ζ_y, without the loss of generality we can set $H = 1$. Let us also recall that the non-trivial minimizers exist only for $c < \sqrt{2}$.

Solving (8.6) with $H = 1$, we obtain a first-order equation

$$\frac{d\zeta}{dy} = \frac{c\zeta\sqrt{(c^2 - 2)\zeta^2 + 2\sqrt{2}\,\zeta - 1}}{\sqrt{2}\,\zeta - 1}, \qquad L_0 < y < L, \tag{8.8}$$

which can be solved implicitly for y. After some tedious algebra, we find (up to an additive constant)

$$y = \frac{1}{c} \left(\frac{\sin^{-1}\left(\frac{(2-c^2)\zeta - \sqrt{2}}{c} \right)}{\sqrt{1 - \frac{c^2}{2}}} - \sin^{-1}\left(\frac{\sqrt{2}\,\zeta - 1}{c\,\zeta} \right) \right), \tag{8.9}$$

where

$$\frac{1}{\sqrt{2}} < \zeta < \frac{1}{\sqrt{2} - c}. \tag{8.10}$$

These limits are chosen from the requirements that $dy/d\zeta = 0$ and $dy/d\zeta = \infty$ at the endpoints of the interval. Now, recalling that $\zeta = e^{cz}$, where $z = h(y)$ is the function whose graph is a minimizer of Π_c, see (5.22), we conclude that we have obtained a parametric representation of this minimizer, once the value of $c = c^\sharp$ is known.

Finally, to find the value of $c^\sharp$, we equate the total variation of y in (8.9) over the interval in (8.10) to $L - L_0$:

$$L - L_0 = \frac{\pi\left(\sqrt{2} - \sqrt{2 - c^2}\right) + 2\sqrt{2}\sin^{-1}\left(\frac{c}{\sqrt{2}}\right)}{2c\sqrt{2 - c^2}} \tag{8.11}$$

The solution of this equation gives $c^\sharp$. We computed the value of $c^\sharp$ for $L = \frac{5}{2}$ numerically and found that $c^\sharp \simeq 1.010$. Therefore, we conclude that for this value of L we have $0.925 \leq c^\dagger \leq 1.011$. Thus, a variational characterization of the traveling wave solutions in the sharp reaction zone limit allowed us to bracket the value of the wave speed within a 5% accuracy, with a minimal computational effort. Also, the curve that minimizes Π_c in this case is shown in Fig. 4(b). Observe the similarity of the main characteristics of the two profiles in Fig. 4, parts (a) and (b).

We now would like to compare these sharp reaction zone limit estimates with the numerical solution of the approximating problem in (6.1). For the purposes of the numerics, we chose the following form of $g(u)$:

$$g(u) = 12u(1-u)^2. \tag{8.12}$$

Fixing $\varepsilon \in (0,1)$, we obtain numerical approximations to the traveling wave solutions on the strip $(0, 2L) \times \mathbb{R}$ with Dirichlet boundary conditions by solving the corresponding parabolic PDE on a sufficiently large rectangle with a localized initial condition (using simple explicit in time, centered in space, finite difference scheme) and waiting sufficient time for an (approximate) traveling wave to form. We find, for example, that when $\varepsilon = 0.2$, the traveling wave has a speed $c^\dagger_\varepsilon \simeq 1.095$. The profile of the wave front for this value of ε is also presented in Fig. 4(c). Note, once again, the similarity between all three profiles in Fig. 4. We also performed a series of simulation in the range $0.1 \leq \varepsilon \leq 0.5$ and extrapolated the value of $c^\dagger_\varepsilon$ to $\varepsilon = 0$, finding $c^\dagger_0 \simeq 0.987$, see Fig. 5, in agreement with the estimates obtained earlier for the sharp reaction zone limit. We note that for $\varepsilon < 0.2$ all three estimates obtained by us are within $\sim$10% of each other. In particular, the value of $c^\sharp$, corresponding to the Markstein model of flame propagation [41], gives a very good approximation for the propagation speed even in the presence of "heat loss" through the walls and curvature comparable to the "flame" size.

Note that from the phase plane analysis it follows that the equilibria of (6.1) are unique for each $\varepsilon > 0$, thus, there exists a unique traveling wave solution for each $\varepsilon \in (0,1)$, which is the minimizer we found. Similarly, the results of Sec. 7 apply for each $\varepsilon \in (0,1)$, and so propagation with speed $c^\dagger_\varepsilon$ is guaranteed for the initial data that approach v_ε as $t \to \infty$ on compacts as $t \to \infty$. In particular, the propagation speed for the parabolic problem will tend to $c^\dagger$ estimated in the first part of this section in the limit $\varepsilon \to 0$.

We conclude this section by presenting a few results for the case when $\varphi \neq 0$. In particular, for $n = 2$ an important special case is that of a linear function $\varphi = \alpha y$, corresponding to a divergence-free flow across the strip. We solved (6.1) numerically with $\alpha = 1$, $L = \frac{5}{2}$, and $\varepsilon = 0.2$, to find a propagation speed $c^\dagger_\varepsilon = 0.698$. The profile of the front in this case is also shown in Fig. 6(a). The value of $c^\dagger_\varepsilon$ is compared with the numerical solution of (8.5) on the domain $\omega_0 = (\log(1+\frac{1}{\sqrt{2}}), 2L+\log(1-\frac{1}{\sqrt{2}}))$. We obtained $c^\sharp \simeq 0.5776$, which, once again, is close to the value of $c^\dagger_\varepsilon$ obtained earlier. Also, the profile of the corresponding minimizer of Π_c is shown in Fig. 6(b). Again, extrapolating the values of $c^\dagger_\varepsilon$

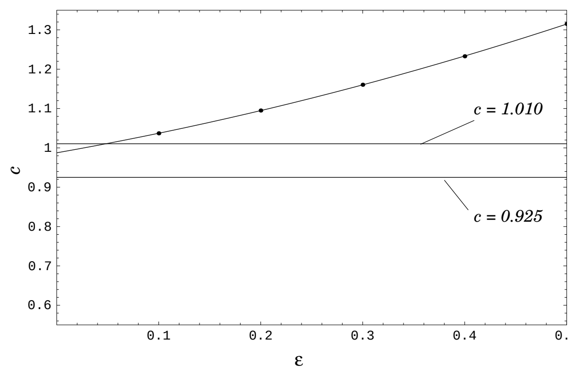

Figure 5: Dependence of $c_\varepsilon^\dagger$ obtained from the numerical solution of (6.1) and (8.12) with $\varphi = 0$. The dots are the results of the simulations, the curve is a fit using a quadratic polynomial $c_\varepsilon^\dagger \simeq 0.987 + 0.458\varepsilon + 0.393\varepsilon^2$. Numerical solutions of (6.1) and the propagation speeds are obtained by solving the associated parabolic problem in (1.1) on a rectangle $(0, 5) \times (0, 20)$ with the initial data $u(y, z) = \cosh^{-2}(\frac{1}{2}\sqrt{(y - \frac{5}{2})^2 + z^2})$ discretized on the 100×400 grid (except for $\varepsilon = 0.1$ when a 200×800 grid was used), with Dirichlet boundary conditions everywhere except at $z = 0$, where Neumann boundary conditions are used.

obtained in the interval $0.1 \le \varepsilon \le 0.5$ to $\varepsilon = 0$ as before, we obtained $c^\dagger \simeq 0.554$ for the sharp reaction zone limit, in agreement with the above upper estimate. To summarize, the value of $c^\sharp$ approximates the value of $c^\dagger$ within 5%, despite the fact that the domain size is comparable with the minimal size in (8.2) for which propagation is possible, and for which the curvature of the front is not small.

Finally, let us illustrate the assumptions of Proposition 5.7 with a numerical example with $L = 10$ and $\varphi_y = -2\cos\left(\frac{\pi y}{2L}\right)$. Since this expression is greater than $\sqrt{2}$ in absolute value outside of the interval $5 \le y \le 15$, the minimizer of Π_c cannot come closer than distance $L/2 \gg 1$ to the boundary, as required by the assumptions of Proposition 5.7. Similarly, since φ_y varies on the length scale of L, the minimizer of Π_c has curvature of order L^{-1}. For this choice of φ, this minimizer is shown in Fig. 7(a). For comparison, Fig. 7(b) shows the numerical solution of (6.1) with $\varepsilon = 0.2$. The value $c_\varepsilon^\dagger \simeq 1.13$ found here is, again, in good agreement with $c^\sharp \simeq 1.016$ found from solving (8.5) numerically. The corresponding extrapolated value $c^\dagger \simeq 0.99$ for $\varepsilon = 0$ limit is, once again, very close to the upper bound. We note that the solution just analyzed is also related to the front solutions found in the edge flame problem (see e.g. [56]), these will be studied in more detail elsewhere.

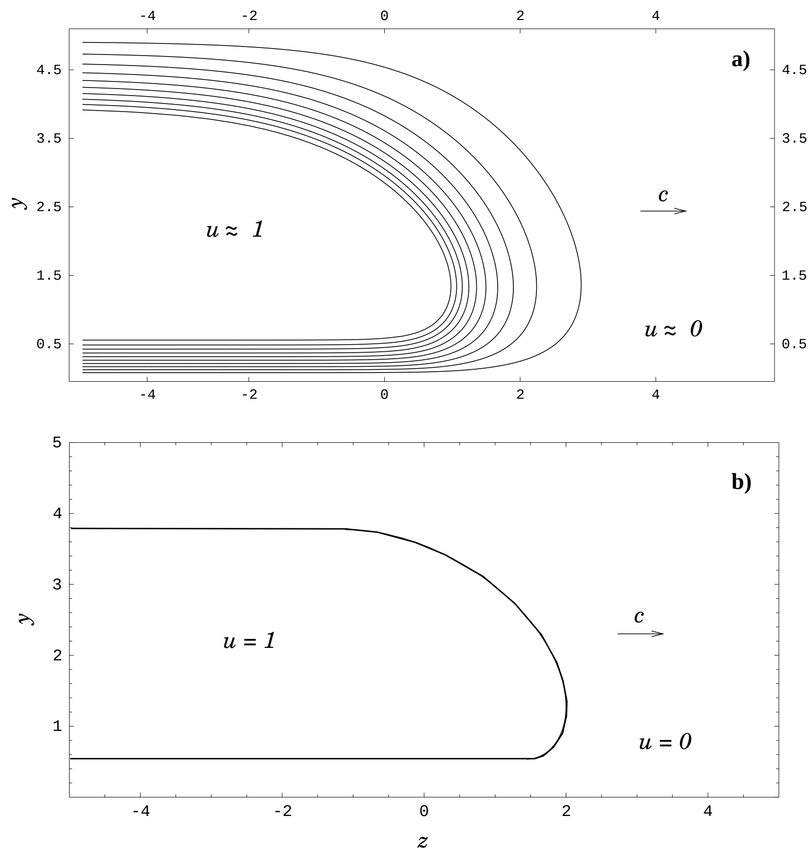

Figure 6: Comparison of the front profiles for $\varphi = y$ and $L = \frac{5}{2}$. (a) The level curves of the numerical solution of (6.1) and (8.12) with $\varepsilon = 0.2$. (b) The curve minimizing Π_c with $c = c^\sharp$.

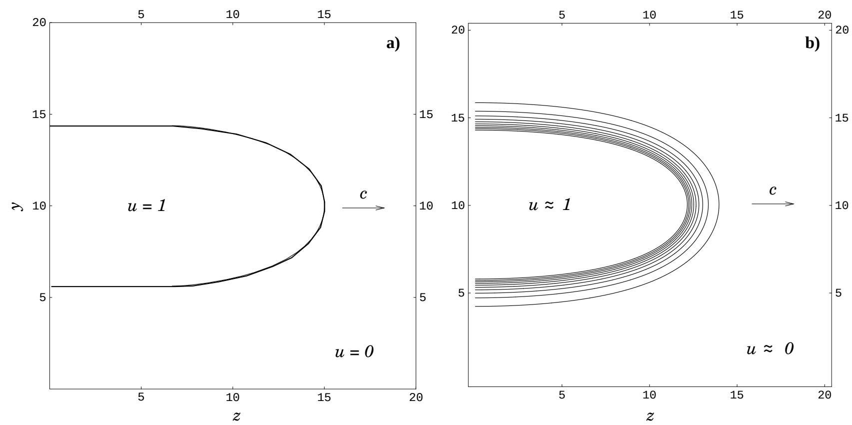

Figure 7: Comparison of the front profiles for $\varphi_y = -2\cos\left(\frac{\pi y}{2L}\right)$ and $L = 10$. (a) The minimizer of Π_c for $c = c^\sharp$. (b) The level curves of the numerical solution of (6.1) and (8.12) with $\varepsilon = 0.2$.

9 Summary and discussion

In this section, we summarize the results obtained in this paper and discuss them in relationship to the current state of the subject. We also point out a number of issues that are currently open.

To recapitulate informally the existence and propagation results in Secs. 4 and 7, we have obtained a rather complete characterization of front propagation for scalar reaction-diffusion-advection problems of the considered type. We found that propagation for the initial value problem with sufficiently localized initial data (e.g. with support bounded at large positive z) cannot occur faster than the speed c^* (see Theorem 7.8), which is either the speed of the minimizer from Theorem 4.3, or the speed of the minimal speed front from Theorem 4.11, if the former does not exist, or zero, if neither of these front solutions exists. Under some extra assumptions we also showed that the speed of these traveling fronts is in fact precisely the asymptotic speed of propagation. Thus, the traveling wave solutions constructed in Sec. 4 play a special role for the propagation phenomena governed by (1.1).

Basically, independently of the details of the nonlinearity f and the potential φ one of the following three scenarios is realized. The choice of the scenario is determined by two ingredients: the minimizers of the functional E in (3.10) and the smallest eigenvalue ν_0 of the self-adjoint operator in (4.4). First, if $v = 0$ is a global minimizer of $E[v]$, then no propagation is possible altogether, in the sense that the leading edge (see (7.3)) of the solution obeys $R_\delta < ct$ asymptotically for large times for any $c > 0$, as follows from Proposition 7.3. So, as is well expected, a necessary condition for propagation is given by (4.45).

If the latter is the case, then two possibilities exist. First, it is possible that the functional Φ_c has minimizers, in that case the speed $c^\dagger$ of Theorem 4.3 is the asymptotic propagation speed for a large class of sufficiently localized initial data, see Propositions 7.3 and 7.5. This covers both the case $\nu_0 > 0$ for which a discrete set of traveling wave solutions (often just one) is expected, and the case $\nu_0 \leq 0$ in which a continuous family of traveling wave solutions is expected [12]. Thus, in the latter case the existence of minimizers provides a selection criterion for the possible propagation speed for sufficiently localized initial data. We note that in the terminology of [54], this corresponds to nonlinear selection, and the minimizers are the so-called "pushed" fronts.

On the other hand, if (4.45) holds and Φ_c has no non-trivial minimizers, then necessarily $\nu_0 < 0$, and the same statements as above hold, in which the propagation speed is now given by c_0, which is determined entirely by the linearization of (4.1) around $u = 0$. This scenario corresponds to the linear selection of [54], and the traveling wave solutions with speed c_0 are the minimal "pulled" fronts. Let us note that these scenarios have already been observed in many particular instances of (1.1). Our results here suggest that the variational characterization of the traveling wave solutions provides a definitive criterion for linear vs. nonlinear selection for a general class of reaction-diffusion-advection problems considered here (see also [37]).

Let us now comment on the relationship of our results with those available in the literature. Equation (1.1) has been studied in an enormous number of works (for references, see Introduction). Let us point out, however, that the main thrust of research on the reaction-diffusion-advection equation in (1.1) has been towards problems with shear flows (e.g. when $\mathbf{v}$ has a z-component which depends on y [12,62,63]). Such problems are motivated by, e.g., considering a Poisueille flow of premixed fuel-oxidizer mixture inside an insulated pipe which can sustain wrinkled deflagration fronts in combustion. We, on the other hand, considered a different setup (for details, see Sec. 2), in which the flow is perpendicular to the cylinder axis. Also, we are constrained to considering only potential flows because of the limitations of our variational approach. So, we cannot readily treat problems of front propagation in shear flows considered in the majority of the literature.

The introduction of a transverse potential flow is a novel aspect of our analysis. Nevertheless, our results can be compared to previous results on existence of traveling wave solutions in the absence of the flow, i.e. for purely reaction-diffusion problems. For Neumann boundary conditions our results naturally extend those of Berestycki and Nirenberg [12] to arbitrary types of nonlinearities and, in particular, to nonlinearities which change type in different portions of the cylinder cross-section. Moreover, under the assumption of uniqueness of the local (hence global) non-degenerate minimizer v of $E[v]$ with negative energy we obtain a unique, monotone, exponentially decaying traveling wave solution which determines the asymptotic propagation speed for a large class of sufficiently localized initial data. This is even true when $\nu_0 = 0$, the case that was left open in the study of [12] and consequent studies. Also, as was already mentioned earlier, in the case $\nu_0 < 0$ existence of minimizers gives a sharp criterion for linear vs. nonlinear selection, i.e. whether $c^* = c_0$ or

$c^* > c_0$ in [12]. In fact, we have proposed a new sufficient condition for linear selection (Proposition 4.10), which is more general than a commonly used assumption $f(u, y) \leq u f_u(0, y)$, which is usually referred to as the KPP-type nonlinearity [9]. Moreover, a more restrictive condition from Remark 4.12 would guarantee both the KPP-type behavior of the traveling waves and the uniqueness of the minimizer of $E[v]$, also for combinations of Neumann and Dirichlet boundary conditions. We also note that the existence of a critical speed c^* in the case $\nu_0 < 0$ which is established by our analysis does not rely on positivity of f any more and, together with Remark 4.12, applies e.g. to nonlinearities like $f(u, y) = u(\mu(y) - u)$ considered in [9].

Our method also easily treats various boundary conditions, in particular, Dirichlet boundary conditions. The papers that are most relevant to our results here are those of Vega [57–59] (see also [28, 32]). Vega constructed the unique solution connecting two stable (in a certain sense) critical points of the functional E, provided there are no other critical points of E with negative energy that are sandwiched between them. He also constructed a family of solutions connecting an unstable equilibrium of E at $z = +\infty$ with the stable one at $z = -\infty$ under a similar assumption on other critical points of E. Our analysis generalizes these results by weakening the assumptions on the nonlinearity, if we redefine the potential in (3.4) to be the negative antiderivative of f only for $0 \leq u(y, z) \leq v(y)$, where v is a stable equilibrium of E with negative energy such that there are no other such equilibria sandwiched between 0 and v. We also only need to verify that $\nu_0 \geq 0$ to ensure existence of a unique, monotone traveling wave solution connecting 0 and v. In the case $\nu_0 < 0$, we obtain existence of the minimal speed front characterized by the fast exponential decay (see also [49]). Let us point out that with our variational approach we are able to obtain various estimates on the traveling wave speed, as well as distinguish between linear and nonlinear selection mechanism. We also similarly can obtain various monotonicity properties of the speed with respect to changes in the nonlinearity or the shape of Ω.

Let us also point out an important limitation of the approach of Vega. Consider a situation in which the domain Ω consists of two sufficiently large mirror-symmetric regions connected by a thin neck, with Dirichlet boundary conditions. Clearly, with a bistable y-independent nonlinearity f (i. e. satisfying (4.45) and $\nu_0 > 0$) one could have three positive local minimizers for the functional E: two that are localized in each of the halves of Ω and one which is localized in both halves. With no other local minima of E one can use the method of Vega to construct the traveling wave solutions which are localized in one of the corresponding halves of the cylinder Σ. Our method, on the other hand, will always pick the (faster) traveling wave solution that is localized in both halves of the cylinder. Indeed, by mirror symmetry the minimizer of Φ_c must also be symmetric, hence the traveling wave will necessarily connect zero with the symmetric local (also global) minimizer of E. In view of the discussion in the previous paragraph, in this situation our method will yield all the traveling wave solutions in the problem after a suitable redefinition of V in each case. We emphasize that in general our method does not rely on the knowledge of the global picture for the critical points of the functional E, in contrast to the approach of Vega.

This brings us to an important question of multiplicity of solutions of (4.1). What we showed in this paper is that the unique traveling wave with speed c^* connects zero to a particular equilibrium v of the functional $E[v]$ (a local minimum with negative energy). With our assumptions on the nonlinearity one would generically expect to have only a finite number of such equilibria, possibly only one (a sufficient condition for this is given in Remark 4.12 which holds for many KPP-type nonlinearities studied in the literature). If there is a unique local minimizer of E with negative energy, the solution we constructed is the unique traveling wave (and has unique speed) with fast exponential decay and gives the propagation speed for all initial data that converge to v on compact sets when $t \to +\infty$, see Corollary 7.9. Actually, in this situation and with $c^* > c_0$, but with a different advection term and for Neumann boundary conditions, together with a number of other assumptions, Roquejoffre was able to prove that the the solution of (1.1) in fact converges to a translate of this traveling wave [49]. It is not clear what happens in a more general situation, when minimizers of E with negative energy are not unique. Clearly, one could e.g. construct other, slower traveling wave solutions which are minimizers of Φ_c with modified potential V that penalizes certain equilibria. Thus, the question of classifying all variational traveling waves and all possible asymptotic propagation speeds in the presence of multiple equilibria of E remains open. Let us point out that simple estimates of [44, Theorem 3.7] can be easily used to exclude certain equilibria as potential candidates for the asymptotic limit of $\bar{u}$ as $z \to -\infty$ for the minimizers of Φ_c.

In short, we have obtained a characterization of propagation in the spirit of Aronson and Weinberger [5] for the considered problem (see also [27]). It would be interesting to see how the notion of propagation related to the motion of the leading edge used here relates to the generalized notion of propagation which was recently introduced for a class of the so-called wave-like solutions in [44, Theorem 4.7]. Let us point out that both definitions of propagation velocity have $c^\dagger$ (or c_0 in the absence of minimizers of Φ_c) as the upper bound. Similarly, the asymptotic propagation speed from [44, Theorem 4.11] gives a lower bound for the propagation speed of the leading edge. One may naturally ask when these two asymptotic propagation speeds are actually the same. We have not yet been able to answer this question. One way to proceed here would be to apply Theorem 4.8 of [44] under the assumption that there are no variational traveling waves other than the minimizer. This, however, seems to be difficult to do, since one needs some a priori information on the exponential decay of the solution of the initial value problem in the reference frame associated with the leading edge. We note that this would also, in turn, imply a much stronger result of convergence of the solution to the initial value problem to a minimizer as $t \to \infty$, in view of the linear stability of the minimizer (by monotonicity, zero is the smallest eigenvalue of the linearization around $\bar{u}$, all other eigenvalues and the essential spectrum are strictly above zero in the weighted space [51]). Alternatively, one could use positivity of Φ_c evaluated on the solution of the initial value problem for $c = \lim_{t\to\infty} \bar{c}(t)$, see [44, Definition 4.5], to interpret Φ_c as a Lyapunov functional for the problem in the reference frame moving with speed c. Here, however, one faces a difficulty associated with the lack of compactness in the problem. In sum, a more precise characterization of propagation in the considered problem is still open.

Finally, let us say a few words about our results in Secs. 5 and 6. The results of Sec. 5 generalize the work of Alt and Caffarelli [2] to the case of transverse potential flows and infinite cylinders. The latter aspect of our analysis is novel and is related to what was done in Sec. 4 to overcome the lack of compactness associated with the translational symmetry in the considered problem. Another novel aspect of our analysis is upper and lower bounds in terms of the minimizers for the associated area-type functional, Propositions 5.6 and 5.7, which give sharp bounds on the propagation speed in terms of the solutions of a mean curvature-type equation (5.31). Also, our convergence analysis of Proposition 6.3 is a counterpart of that of Berestycki, Caffarelli and Nirenberg [7] in a different context. Moreover, we obtained stronger convergence results than in [7], and explicit estimates on the propagation speed for the regularized problem in terms of that of the free boundary. Let us note that, similarly to the regular case, the question of multiplicity of the traveling wave solutions in Theorem 5.1 has to do with multiplicity of local minimizers with negative energy of the functional E_0 in (5.3). We conclude by noting that in practice our variational formulations can give very good numerical estimates for the propagation speed in combustion problems, as was shown in Sec. 8.

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