NULL LAGRANGIAN MEASURES IN PLANES, COMPENSATED COMPACTNESS AND CONSERVATION LAWS

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Abstract. Compensated compactness is an important method used to solve nonlinear PDEs. It is particularly successful in the study of hyperbolic conservation laws. One of the simplest formulations of a compensated compactness problem is to ask for conditions on a set $\mathcal{K} \subset M^{m\times n}$ such that

$$\lim_{n\to\infty} \mathrm{dist}(Du_n, \mathcal{K}) \overset{L^p}{\to} 0 \ \Rightarrow \{Du_n\}_n \text{ is precompact.} \tag{1}$$

Let $M_1, M_2, \dots, M_q$ denote the set of all minors of $M^{m\times n}$. A sufficient condition for (1) is that any measure μ supported on $\mathcal{K}$ satisfying

$$\int M_k(X) d\mu(X) = M_k\left(\int X d\mu(X)\right) \text{ for } k = 1, 2, \dots, q \tag{2}$$

is a Dirac measure. We call measures that satisfy (2) *Null Lagrangian Measures* and following [Mü 99] we denote the set of Null Lagrangian Measures supported on $\mathcal{K}$ by $\mathcal{M}^{pc}(\mathcal{K})$. For general m, n, a necessary and sufficient condition for triviality of $\mathcal{M}^{pc}(\mathcal{K})$ is unknown even in the case where $\mathcal{K}$ is a linear subspace of $M^{m\times n}$. We provide a condition that is sufficient for any linear subspace $\mathcal{K}$ and also necessary in the case where $2 \leq \min\{m, n\} \leq 3$. A corollary of this result is the fact that two dimensional subspaces $\mathcal{K} \subset M^{m\times n}$ support nontrivial Null Lagrangian Measures if and only if $\mathcal{K}$ has Rank-1 connections.

Using the ideas developed for this purpose we are able to answer (up to first order) a question of Kirchheim, Müller and Šverák [Ki-Mü-Sv 03]. Let $P_1(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix}$ and $\mathcal{K}_1 := \{P_1(u,v) : u, v \in \mathbb{R}\}$ for some function a and its primitive F. The set $\mathcal{K}_1$ arises in the study of entropy solutions to the 2×2 system of conservation laws

$$u_t = a(v)_x \quad \text{and} \quad v_t = u_x. \tag{3}$$

In [Ki-Mü-Sv 03], the authors asked what the conditions on the function a are such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap U)$ consists of Dirac measures, where U is an open neighborhood of an arbitrary matrix in $\mathcal{K}_1$. Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $a'(\alpha_2) > 0$ then we construct nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\alpha)))$ for any $\delta > 0$. On the other hand if $a'(\alpha_2) < 0$ then for sufficiently small $\delta > 0$, we show that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\alpha)))$ consists of Dirac measures.

Further exploiting these ideas leads to a strategy of proving trivially of Null Lagrangian Measures supported on a small neighborhood of a smooth submanifold. We use this strategy to provide a more direct proof of DiPerna's well known result [DP 85] on existence of entropy solutions to (3). The strategy could potentially be applied to other systems of conservation laws with few entropies.

1. Introduction

Compensated compactness is an important method of solving nonlinear PDEs. Amongst its most celebrated successes are the proofs of the first existence theorems for solutions of systems of hyperbolic conservation laws with large data by Tartar [Ta 79], [Ta 83] and DiPerna [DP 83], [DP 85]. One of the simplest and most natural formulations of compensated compactness is to ask for conditions on a set of matrices $\mathcal{K} \subset M^{m\times n}$ such that for any sequence

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$\{u_k\} \subset W^{1,p}_{loc}(\Omega;\mathbb{R}^m)$ defined on a domain $\Omega \subset \mathbb{R}^n$, if

$$\mathrm{dist}(Du_k,\mathcal{K}) \to 0 \text{ strongly in } L^p(\Omega) \quad \text{and} \quad u_k \overset{W^{1,p}_{loc}(\Omega)}{\rightharpoonup} u \text{ as } k \to \infty, \tag{4}$$

then

$$Du_k \overset{L^p(\Omega)}{\to} Du \text{ as } k \to \infty. \tag{5}$$

It turns out that necessary and sufficient conditions on $\mathcal{K}$ for hypothesis (4) to imply (5) is the following:

For any probability measure μ with $\mathrm{Spt}\mu \subset \mathcal{K}$,

$$\int f(X)d\mu(X) = f\left(\int X d\mu(X)\right) \text{ for all Quasiconvex functions } f \implies \mu \text{ is a Dirac measure.}$$

This follows from Theorem 4.7 in [Mü 99] (see [Ki-Pe 91] for the original source) and the fundamental theorem of Young measures (Theorem 3.1 and Corollary 3.2 in [Mü 99]). However Quasiconvex functions are very hard to understand [1], so more commonly a smaller class of functions known as *Polyconvex* functions are considered. These functions were introduced by Ball [Ba 77] in his fundamental work on existence of minimizers of elasticity functionals. Given $X \in M^{m\times n}$, let $\hat{X}$ denote the vector of all minors of X. A polyconvex function is a function $f : M^{m\times n} \to \mathbb{R}$ that can be written as $f(X) = g(\hat{X})$ where g is convex.

Following [Mü 99], given $\mathcal{K} \subset M^{m\times n}$, we denote

$$\mathcal{M}^{pc}(\mathcal{K}) := \left\{ \begin{array}{cc} \nu \in \mathcal{P}(M^{m\times n}) : & \mathrm{Spt}(\nu) \subset \mathcal{K}, \int f(X)d\nu(X) = f\left(\overline{X}\right) \text{ for all} \\ & \text{polyconvex functions } f, \text{where } \overline{X} = \int X d\nu(X) \end{array} \right\}.$$

A function $g : M^{m\times n} \to \mathbb{R}$ is a *Null Lagrangian* if $g(X)$ is an affine combination of the minors of $X \in M^{m\times n}$. Clearly if g is a *Null Lagrangian* then both g and $-g$ are polyconvex. Therefore $\mu \in \mathcal{M}^{pc}(\mathcal{K})$ if and only if

$$\int M(X)d\mu(X) = M\left(\int X d\mu(X)\right) \text{ for all minors } M.$$

For this reason we shall call measures $\mu \in \mathcal{M}^{pc}(\mathcal{K})$ *Null Lagrangian Measures*.

As we will briefly sketch, the heart of a number of well known compensated compactness results is a proof that for some submanifold $\mathcal{K}$ in the space of matrices, $\mathcal{M}^{pc}(\mathcal{K})$ consists of Dirac measures. There is overall little understanding of what general conditions a set $\mathcal{K}$ has to have in order for $\mathcal{M}^{pc}(\mathcal{K})$ to consist of Dirac measures only, i.e., to be trivial. Even in the case when $\mathcal{K}$ is a linear subspace in the space of matrices, it is an open problem to determine necessary and sufficient conditions on $\mathcal{K}$ for $\mathcal{M}^{pc}(\mathcal{K})$ to be trivial [2]. One of our main results provides such sufficient condition for subspaces in $M^{m\times n}$ that is necessary in the case where $2 \le \min\{m,n\} \le 3$. This is a corollary to the following Theorem 1.

First we require some notations. Given a set of homogeneous polynomials

$$\mathcal{F} := \{f_k : \mathbb{R}^n \to \mathbb{R} : k = 1, 2, \dots, q\},$$

we define the set of Null Lagrangian Measures with respect to the set of polynomials $\mathcal{F}$ by

$$\mathbb{M}^{pc}_{\mathcal{F}} := \left\{\mu \in \mathcal{P}(\mathbb{R}^n) : f\left(\int z\, d\mu(z)\right) = \int f(z)\, d\mu(z) \text{ for all } f \in \mathcal{F}\right\} \tag{6}$$

and further we define $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi) := \left\{\mu \in \mathbb{M}^{pc}_{\mathcal{F}} : \int z\, d\mu(z) = \varpi\right\}$.

[1] Indeed one of the most important problems in Calculus of Variations is the question of whether in 2×2 matrices, Rank-1 convex functions are Quasiconvex [Ba 85], [Ast 98].

[2] This was asked to the first author by V. Šverák during a brief sabbatical visit to Minnesota in 2016.

Theorem 1. *Let $f_1, f_2, \ldots, f_{M_1} : \mathbb{R}^n \to \mathbb{R}$ be homogeneous polynomials with the property that*

$$2 \leq \deg(f_k) \leq 3 \text{ for } k = 1, 2, \ldots, M_1,$$

where we have ordered them such that $\deg(f_k) = 2$ for $k = 1, 2, \ldots, M_0$ and $\deg(f_k) = 3$ for $k = M_0 + 1, M_0 + 2, \ldots, M_1$. Let $\mathcal{F} := \{f_1, f_2, \ldots, f_{M_1}\}$. Then

$$\mathbb{M}^{pc}_{\mathcal{F}} \text{ consists of Dirac measures}$$

if and only if the following condition holds:

$$\text{For each subspace } V \subset \mathbb{R}^n \text{ there exists } \beta \in S^{M_0-1} \text{ such that}$$
$$\sum_{k=1}^{M_0} \beta_k f_k(z) \geq 0 \text{ for all } z \in V \text{ and } \sum_{k=1}^{M_0} \beta_k f_k \not\equiv 0 \text{ on } V. \tag{7}$$

Further if V is a subspace for which (7) fails, then for any $\varpi \in V$ there exists a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$.

As a consequence, we have

Corollary 2. *Suppose $K \subset M^{m\times n}$ is a linear subspace. Let $M_1, \ldots, M_{q_0} : M^{m\times n} \to \mathbb{R}$ be the set of all nontrivial 2×2 minors on K. If*

$$\text{for each subspace } L \subset K \text{ there exists } \beta \in S^{q_0-1} \text{ such that}$$
$$\sum_{k=1}^{q_0} \beta_k M_k(X) \geq 0 \text{ for all } X \in L \text{ and } \sum_{k=1}^{q_0} \beta_k M_k \not\equiv 0 \text{ on } L, \tag{8}$$

then

$$\mathcal{M}^{pc}(K) \text{ consists of Dirac measures.} \tag{9}$$

Conversely if $2 \leq \min\{m, n\} \leq 3$ then (9) implies (8).

Remark 1. We say that a set $\Sigma \subset M^{m\times n}$ has *Rank*-1 *connections* if and only if there exist $A, B \in \Sigma$ such that $A \neq B$ and $\text{Rank}(A - B) = 1$. Note that if $K \subset M^{m\times n}$ is a subspace that satisfies (8), then K has no Rank-1 connections. Indeed, were this not the case, there would be a nontrivial subspace $L \subset K$ with the property that $\text{Rank}(A) = 1$ for every $A \in L$, and therefore $M_k(A) = 0$ for every $k = 1, 2, \ldots, q_0$, which contradicts condition (8). It could be that for subspaces, condition (8) is actually equivalent to having no Rank-1 connections. We will present a proof of this for two dimensional subspaces in $M^{m\times n}$ in Lemma 12 of Section 5. As a consequence of Lemma 12 and Corollary 2, we have the following attractive consequence:

Corollary 3. *Let $K \subset M^{m\times n}$ be a two dimensional subspace, then $\mathcal{M}^{pc}(K)$ consists of Diracs if and only if K does not contain Rank-1 connections.*

Note that in [Sv 93], Šverák proved the beautiful result that for connected sets $\mathcal{K} \subset M^{2\times 2}$, $\mathcal{M}^{pc}(\mathcal{K})$ is trivial if and only if $\mathcal{K}$ does not contain Rank-1 connections. This result is false for general $M^{m\times n}$ (see Section 2.5 in [Mü 99]), however it might be true for subspaces. This would be a very attractive result, however for the applications that we have developed in this paper, condition (8) is actually more useful and informative.

One of the motivations for studying Null Lagrangian Measures supported on planes is that such results might rather directly yield insights into how to prove triviality of $\mathcal{M}^{pc}(\mathcal{K} \cap U)$ where $\mathcal{K}$ is any smooth submanifold and U is a small neighborhood around an arbitrary point in $\mathcal{K}$. In the following subsection we apply these insights to study a well known 2×2 system of conservation laws.

1.1. **Connections and applications to conservation laws.** As mentioned above one of the main triumphs of compensated compactness is the proof of existence theorems for hyperbolic conservation laws. To sketch this briefly, the standard way to solve a scalar equation is to add a viscosity term and obtain a solution to

$$u^\epsilon_t + G(u^\epsilon)_x = \epsilon u^\epsilon_{xx} \text{ in } (0,\infty)\times\mathbb{R}. \tag{10}$$

Assuming $\{u^\epsilon\}_\epsilon$ is bounded in $L^\infty((0,\infty)\times\mathbb{R})$ we can extract a subsequence $u^{\epsilon_k} \overset{*}{\rightharpoonup} u$ in $L^\infty((0,\infty)\times\mathbb{R})$. Letting $\nu_{t,x}$ be the Young measure associated with the weak* convergence, i.e., $u(t,x) = \int_{\mathbb{R}} y\, d\nu_{t,x}$, we have $G(u^{\epsilon_k}) \overset{*}{\rightharpoonup} \overline{G}$ in $L^\infty((0,\infty)\times\mathbb{R})$ where $\overline{G}(t,x) = \int G(y)\, d\nu_{t,x}$.

Now for any convex function $\Phi:\mathbb{R}\to\mathbb{R}$, define $\Psi(y) := \int_0^y \Phi'(s)G'(s)ds$. The pair (Φ,Ψ) is called an entropy/entropy flux pair. The key point is that by virtue of the Div-Curl lemma we know that

$$\int (G(y)\Phi(y) - y\Psi(y))\; d\nu_{t,x} = \overline{G}(t,x)\overline{\Phi}(t,x) - u(t,x)\overline{\Psi}(t,x), \tag{11}$$

where $\overline{\Phi}(t,x) = \int \Phi(y)d\nu_{t,x}$ and $\overline{\Psi}(t,x) = \int \Psi(y)d\nu_{t,x}$. Define $P_\Phi:\mathbb{R}\to M^{2\times 2}$ by $P_\Phi(z) := \begin{pmatrix} G(z) & z \\ \Psi(z) & \Phi(z)\end{pmatrix}$ and the measure μ_Φ on the set $\mathcal{K}_\Phi := \{P_\Phi(z) : z\in\mathbb{R}\}$ by $\mu_\Phi := (P_\Phi)_\sharp \nu_{t,x}$, the push forward of $\nu_{t,x}$ by the map P_Φ. By (11), $\mu_\Phi \in \mathcal{M}^{pc}(\mathcal{K}_\Phi)$. So to prove triviality of $\nu_{t,x}$ it suffices to prove $\mathcal{M}^{pc}(\mathcal{K}_\Phi)$ is trivial for any choice of convex function Φ. As this is such a wide class, for a lot of scalar conservation laws, one can find an appropriate convex function Φ for which $\mathcal{M}^{pc}(\mathcal{K}_\Phi)$ is trivial, and hence the Young measures are trivial. The fact that u is a weak solution to (10) without viscosity term follows from triviality of Young measures in a standard way.

For systems of conservation laws, however, a key point is that for many systems there are only finitely many entropy/entropy flux pairs (Φ_1,Ψ_1), (Φ_2,Ψ_2), $\ldots$, (Φ_m,Ψ_m). By analogous argument to the scaler case, triviality of Young measures would follow from triviality of $\mathcal{M}^{pc}(\mathcal{K}_{\Phi_1})$, $\mathcal{M}^{pc}(\mathcal{K}_{\Phi_2})$, $\ldots$, $\mathcal{M}^{pc}(\mathcal{K}_{\Phi_m})$. For 2×2 systems we can formulate this by considering a single submanifold of matrices $\mathcal{K}$ whose rows are given by the the rows of $\mathcal{K}_{\Phi_1}$, $\mathcal{K}_{\Phi_2}$, $\ldots$, $\mathcal{K}_{\Phi_m}$ all stacked together without repeating the 2×2 equations, then trivially of the Young measures (and hence proof of existence of solutions via compensated compactness) comes down to proving $\mathcal{M}^{pc}(\mathcal{K})$ is trivial.

One of the best known results in this area is the work of DiPerna [DP 85] on the Lagrangian equations of elasticity given by

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0. \end{cases} \tag{12}$$

Here the function $a:\mathbb{R}\to\mathbb{R}$ is a smooth, strictly convex and increasing function. Following [DP 85] we introduce the natural entropy/entropy flux pair (η_1,q_1), and its dual pair (η_2,q_2), associated to the system (12). More precisely, we define

$$\eta_1(u,v) := \frac{1}{2}u^2 + F(v), \quad q_1(u,v) := -ua(v), \tag{13}$$

and

$$\eta_2(u,v) := uv, \quad q_2(u,v) := -\frac{1}{2}u^2 - \tau(v), \tag{14}$$

where $F(\xi) = \int_0^\xi a(s)ds$ and τ is related to the Legendre transform of F. Formally $\tau(p) := \tilde{\tau}\,(a(p))$ where $\tilde{\tau}$ is the Legendre transform of F. Thus

$$\tau(p) = a(p)p - F(p), \tag{15}$$

see (155) of Lemma 23 in the Appendix. Note that this is not how DiPerna [DP 85] defined q_2. Indeed he defined $q_2(u,v) := \frac{u^2}{2} + \tilde{\tau}(v)$. However with q_2 defined this way, the pair (η_2, q_2) does not form an entropy/entropy flux pair [3].

As in [Ki-Mü-Sv 03], we consider entropy solutions of (12) defined as L^∞ functions (u,v) satisfying either

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0, \\ (\eta_1)_t + (q_1)_x \leq 0 \end{cases} \tag{16}$$

or

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0, \\ (\eta_1)_t + (q_1)_x \leq 0, \\ (\eta_2)_t + (q_2)_x \leq 0 \end{cases} \tag{17}$$

in the sense of distributions. Adding a viscosity term to the first two equations of (16), (17), we obtain the pair (u^ϵ, v^ϵ) that solves

$$v_t^\epsilon - u_x^\epsilon = \epsilon v_{xx}^\epsilon, \;\; u_t^\epsilon - a\left(v^\epsilon\right)_x = \epsilon u_{xx}^\epsilon.$$

Assuming appropriate bounds on $u^\epsilon, v^\epsilon, u_x^\epsilon, v_x^\epsilon$ (see (5.38) of [Ev 90]), we obtain the system (16) or (17) with right hand side precompact in $W_{loc}^{-1,2}$. Hence as we have sketched for scalar equations, we have $(u^\epsilon, v^\epsilon) \overset{*}{\rightharpoonup} (u,v)$ in L^∞ and the Young measure can be pushed forward into the set $\mathcal{K}_1$ or $\mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined to be

$$\mathcal{K}_1 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix} : u, v \in \mathbb{R} \right\}, \tag{18}$$

and

$$\mathcal{K}_2 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \\ \frac{1}{2}u^2 + \tau(v) & uv \end{pmatrix} : u, v \in \mathbb{R} \right\}, \tag{19}$$

respectively. By use of the Div-Curl lemma we have a measure in $\mathcal{M}^{pc}(\mathcal{K}_1)$ or $\mathcal{M}^{pc}(\mathcal{K}_2)$. Thus the existence of solutions of the system (12) would follow from triviality of $\mathcal{M}^{pc}(\mathcal{K}_1)$ or $\mathcal{M}^{pc}(\mathcal{K}_2)$ (clearly triviality of $\mathcal{M}^{pc}(\mathcal{K}_1)$ would imply triviality $\mathcal{M}^{pc}(\mathcal{K}_2)$).

1.1.1. *Nontrivial Null Lagrangian Measures for the system with one entropy.* Understanding the structure of $\mathcal{K}_1$ or $\mathcal{K}_2$ plays a fundamental role in understanding the system (16) or (17). Roughly speaking, if there is so little rigidity of the structure of $\mathcal{K}_1$ that certain subset $\mathcal{K}_1^{rc}$ of $\mathcal{K}_1^{pc}$ ($\mathcal{K}_1^{pc}$ and $\mathcal{K}_1^{rc}$ are called the Polyconvex hull and Rank-1 convex hull of $\mathcal{K}_1$, respectively, see Section 4.4 in [Mü 99]) is sufficiently nontrivial, then a very different kind of nontrivial solution to (16) can be obtained as a differential inclusion into $\mathcal{K}_1$[4]. There have been enormous interests and specular progresses in reformulating PDEs as differential inclusions and obtaining solutions via *convex integration* [De-Sz 09], [De-Sz 13], [Bu-De-Is-Sz 15], [Is 17]. Some of the initial impetus for these works come from the pioneering work on Calculus of Variations by [Mü-Sv 96], [Mü-Sv 03], [Mü-Sy 01], [Ki 01], [Ki 03] and the initial ideas come from the work on isometric immersions of Nash [Na 54], Kuiper [Ku 55] and Gromov [Gr 86].

[3] Further Equations (9.8) of [DP 85] do not hold true if $q_2(u,v) = \frac{u^2}{2} + \tilde{\tau}(v)$. However with q_2 defined as in (14), (η_2, q_2) does form an entropy/entropy flux pair and Equations (9.8) of [DP 85] do hold true, see Lemma 23 in the Appendix.

[4] Note that a differential inclusion into set $\mathcal{K}_1$ gives a solution to (16) with the inequality replaced by an equality.

Note that these solutions are wildly non-unique and constructed to have many strange and non-physical properties. The solutions obtainable through vanishing viscosity and compensated compactness are good (in the scalar case) and reasonable solutions in the sense that they are unique and have BV regularity. In contrast, the solutions obtained via convex integration are wild solutions whose existence essentially implies that the problem requires more assumptions to be reasonably posed.

For a set of a matrices $\mathcal{K}$, the notions of *Rank*-1 *convex hull* $\mathcal{K}^{rc}$, *Quasiconvex hull* $\mathcal{K}^{qc}$ and *Polyconvex hull* $\mathcal{K}^{pc}$ can be found, for example, in [Mü 99] (see also [Ki-Pe 91, Sv 95]). Roughly speaking if the Rank-1 convex hull is nontrivial enough, an enormous class of Lipschitz solutions can be constructed with many unusual properties. Since *Polyconvexity* $\Rightarrow$ *Quasiconvexity* $\Rightarrow$ *Rank*-1 *convexity* , we have $\mathcal{K}^{rc} \subset \mathcal{K}^{qc} \subset \mathcal{K}^{pc}$. As mentioned before, Quasiconvex functions are hard to understand and so a first step to proving non-triviality of $\mathcal{K}^{rc}$ is to show non-triviality of $\mathcal{K}^{pc}$. Further as $\mathcal{K}^{pc} = \left\{ \int X d\mu(X) : \mu \in \mathcal{M}^{pc}(\mathcal{K}) \right\}$ (see Exercise 1, Section 4.4 in [Mü 99]), a first step towards this goal is to show non-triviality of $\mathcal{M}^{pc}(\mathcal{K})$. For this reason Kirchheim, Müller and Šverák [Ki-Mü-Sv 03] asked the following question with respect to system (16) and its associated differential inclusion into the set $\mathcal{K}_1$: what are the natural assumptions on the function a such that the following statement is true:

(S1) For each point $\zeta \in \mathcal{K}_1$, there exists a neighborhood $U \subset M^{3\times 2}$ of ζ such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap \overline{U})$ is trivial.

For the system (12) without implementing any entropy/entropy flux pairs, the statement **(S1)** for the corresponding set $\mathcal{K}_0 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \end{pmatrix} : u, v \in \mathbb{R} \right\}$ is well understood using results in [Sv 93]. On the other hand, it is proved in [DP 85] that a set closely related to $\mathcal{K}_2$ (see Subsection 1.1.2 for more details) associated to the system (17) with two entropy/entropy flux pairs satisfies statement **(S1)** when the function a satisfies $a' > 0$ and $a'' \neq 0$. However, this question (as well as some other related properties) for the set $\mathcal{K}_1$ defined in (18) (for the system (16) with just one entropy/entropy flux pair) remained open. (For more details, see [Ki-Mü-Sv 03], Section 7.)

For the convenience of later discussions, we parametrize the sets $\mathcal{K}_1$ and $\mathcal{K}_2$ by the mappings

$$P_1(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix} \tag{20}$$

and, recalling $\tau(v) \overset{(15)}{=} va(v) - F(v)$,

$$P_2(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \\ \frac{1}{2}u^2 + va(v) - F(v) & uv \end{pmatrix}. \tag{21}$$

In this notation, $\mathcal{K}_1 = \{P_1(u,v) : u, v \in \mathbb{R}\}$ and $\mathcal{K}_2 = \{P_2(u,v) : u, v \in \mathbb{R}\}$. In Section 7, given a point $P_1(\tilde{\alpha}) \in \mathcal{K}_1$, we will show that statement **(S1)** is false if $a'(\tilde{\alpha}_2) > 0$ and true if $a'(\tilde{\alpha}_2) < 0$. Specifically, we have

Theorem 4. *Suppose $a \in C^2(\mathbb{R})$. Given $\tilde{\alpha} \in \mathbb{R}^2$, if $a'(\tilde{\alpha}_2) > 0$, then there exist nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\tilde{\alpha})))$ for all $\delta > 0$. On the other hand, if $a'(\tilde{\alpha}_2) < 0$, then there exists $\delta_0 > 0$ depending on the function a and $\tilde{\alpha}_2$ such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\tilde{\alpha})))$ is trivial for all $0 < \delta \le \delta_0$.*

Indeed the second part of Theorem 4 can be made a bit stronger. More precisely, recall that $\mathcal{K}_0 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \end{pmatrix} : u, v \in \mathbb{R} \right\}$. Given $\tilde{\alpha} \in \mathbb{R}^2$, if $a'(\tilde{\alpha}_2) < 0$ then $\mathcal{M}^{pc}\left(\mathcal{K}_0 \cap B_\delta\left(\begin{pmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ a(\tilde{\alpha}_2) & \tilde{\alpha}_1 \end{pmatrix}\right)\right)$ is trivial for sufficiently small $\delta > 0$ depending on a and $\tilde{\alpha}_2$. As $\mathcal{M}^{pc}(\mathcal{K}_1)$ can be naturally embedded into $\mathcal{M}^{pc}(\mathcal{K}_0)$, this implies the second part of the theorem (see the proof of Theorem 4 in Section 7). Theorem 4 is closely related to Theorem 1. Indeed, one can check directly that for the submanifold $\mathcal{K}_1$ given in (18), there does not exist nontrivial linear combination of all three minors that remains nonnegative. Nevertheless, it should be noted that the set $\mathcal{K}_1$ given in (18) is a nonlinear submanifold in the space of 3×2 matrices whose nonlinear structure poses extremely delicate issues. As a result, the arguments needed are significantly beyond those used for linear spaces. The proof of the first part in Theorem 4 uses a construction of discrete probability measures supported at five points with certain symmetry structure. A key ingredient is to place the majority of the mass of the measure at one point in $\mathcal{K}_1$, and think of the other four points as small perturbations. This way one can manage to handle the nonlinear effect of the submanifold $\mathcal{K}_1$ as a small perturbation of the corresponding linear structure, and as a result, the ideas used in the linear cases can be generalized. Moreover, our construction allows to produce infinitely many elements in $\mathcal{M}^{pc}(\mathcal{K}_1)$ (see Theorem 18).

1.1.2. *A direct proof of DiPerna's theorem for the system with two entropies.* As sketched in Subsection 1.1, if there is enough "rigidity of structure" in the set $\mathcal{K}_1$ or $\mathcal{K}_2$ to force the Null Lagrangian Measures on them to be Dirac measures, we obtain solutions from the vanishing viscosity approximations via compensated compactness. This motivated the work of DiPerna on existence of weak solutions to the system (12). In [DP 85], the following theorem is proved:

Theorem 5 (DiPerna [DP 85])**.** *Suppose a is a smooth function with $a' > 0$ and $a'' \neq 0$. Let (η_1, q_1) and (η_2, q_2) be defined by (13) and (14). Assume that $\{(u^\epsilon, v^\epsilon)\}$ is a sequence with small oscillations such that the sequences*

$$u^\epsilon_t - (a(v^\epsilon))_x, \quad v^\epsilon_t - u^\epsilon_x, \quad (\eta_i(u^\epsilon, v^\epsilon))_t + (q_i(u^\epsilon, v^\epsilon))_x$$

are precompact in $W^{-1,2}_{loc}(\mathbb{R}^2)$, $i = 1, 2$. Then there exists a subsequence $\{(u^{\epsilon_n}, v^{\epsilon_n})\}$ that converges to some (u, v) in the strong topology of $L^1_{loc}(\mathbb{R}^2)$.

The way that DiPerna proves Theorem 5 relies on using the entropy/entropy flux pairs (η_i, q_i) to show triviality of Young measures associated to the system. This in turn is closely related to triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$. However, strictly speaking DiPerna did not prove trivially of $\mathcal{M}^{pc}(\mathcal{K}_2)$. Rather he showed trivially of Null Lagrangian Measures on a set $\mathcal{K}_2^\alpha$ (see (74)). The set $\mathcal{K}_2^\alpha$ is the analogue of $\mathcal{K}_2$ with the entropy/entropy flux pairs $(\eta_1, q_1), (\eta_2, q_2)$ replaced by simplified pairs $(Q_\alpha \eta_1, Q^*_\alpha q_1)$, $(Q_\alpha \eta_2, Q^*_\alpha q_2)$ given by (148) and (149). Indeed, the sets $\mathcal{M}^{pc}(\mathcal{K}_2)$ and $\mathcal{M}^{pc}(\mathcal{K}_2^\alpha)$ are equivalent. This fact was implicitly stated in [Ki-Mü-Sv 03], see Lemma 15 in Section 6 for the precise statement and for the proof [5]. In this paper, we provide a direct proof of triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$ (and also $\mathcal{M}^{pc}(\mathcal{K}_2^\alpha)$) motivated by the perspective of searching for linear combinations of minors that are positive. More precisely, we provide a direct proof of

Theorem 6 (DiPerna [DP 85])**.** *Suppose $a \in C^\infty(\mathbb{R})$. Given $\tilde{\alpha} \in \mathbb{R}^2$ such that $a'(\tilde{\alpha}_2) > 0$ and $a''(\tilde{\alpha}_2) \neq 0$, there exists $\delta_0 > 0$ depending on the function a and $\tilde{\alpha}_2$ such that $\mathcal{M}^{pc}(\mathcal{K}_2 \cap B_\delta(P_2(\tilde{\alpha})))$ is trivial for all $0 < \delta \leq \delta_0$.* [6]

As a simple consequence of Theorem 6, we will give a proof of DiPerna's Theorem 5. The perspective we have learned from studying Null Lagrangian Measures supported on planes L is that triviality of Null Lagrangian Measures follows if (and in lower dimensions only if) we can find a linear combination of minors $\sum \lambda_{ij} M_{ij}$ such that $\sum \lambda_{ij} M_{ij}(X)$ is nonnegative on

[5]The proof included was provided to us by S. Müller.

[6]The fact that DiPerna's theorem can be formulated this way is due to [Ki-Mü-Sv 03], as such some credit for Theorem 6 is due to these authors.

L and hence in low dimensions is convex. This does not hold true if we have a subset of matrices $\mathcal{K}$ where $\mathcal{K}$ is not a plane, the main issue being that the barycenter of a measure supported on $\mathcal{K}$ needs not to be on $\mathcal{K}$. However if $\mathcal{K}$ is very close to a plane, then the barycenter will be very close to being on $\mathcal{K}$. So if we can find a linear combination $\sum \lambda_{ij} M_{ij}$ such that the Taylor expansion of

$$X \to \sum \lambda_{ij} M_{ij}(X - X_0) \tag{22}$$

(up to some order) is convex on $\mathcal{K} \cap B_\delta(X_0)$, then there is a very good chance that we can prove triviality of $\mathcal{M}^{pc}(\mathcal{K} \cap B_\delta(X_0))$. Specifically, given $\mu \in \mathcal{M}^{pc}(\mathcal{K})$, if the Taylor expansion of the mapping in (22) is convex, then letting $\overline{X} = \int X d\mu(X)$ we have that

$$\int \sum \lambda_{ij} M_{ij}(X - X_0)\, d\mu(X) = \sum \lambda_{ij} M_{ij}(\overline{X} - X_0). \tag{23}$$

Now if $\overline{X}$ was actually on $\mathcal{K}$ then this would contradict Jensen's inequality unless μ is a Dirac measure. In general $\overline{X} \notin \mathcal{K}$, but since locally $\mathcal{K}$ is very close to a plane, $\overline{X}$ is very close to $\mathcal{K}$ and so $\sum \lambda_{ij} M_{ij}(\overline{X} - X_0)$ can be estimated. If it can be shown to be sufficiently small, then the error can be absorbed into the left hand side of (23) and hence μ can be shown to be a Dirac measure.

DiPerna's proof in [DP 85] consists of manipulating the equations of the form

$$\int M_{ij}(X - X_0) d\mu(X) = M_{ij}(\overline{X} - X_0),$$

or more specifically their Taylor expansions up to a certain order. After a sequence of substitutions DiPerna obtains an inequality that contradicts Jensen's inequality unless μ is a Dirac measure. However the method outlined in the paragraph above is a *strategy*. In Section 8 we provide a proof of DiPerna's theorem by following this strategy.

Remark 2. In the case $a'(\tilde{\alpha}_2) < 0$, the conclusion of Theorem 6 still holds true. This is clear from the proof of Theorem 4 at the end of Section 7.

Remark 3. In principle the strategy outlined above can be applied to other systems of conservation laws as well as other nonlinear PDE problems. One particular potential application involves a classical problem in Calculus of Variations, called the Aviles-Giga functional. This problem shares common features with scalar conservation laws as observed in [De-Mü-Ko-Ot 01], and the tool of entropies has been used intensively in the study of this problem [Ji-Ko 00, Am-De-Ma 99, De-Mü-Ko-Ot 01]. In particular, using compensated compactness arguments involving infinitely many entropies, the authors in [De-Mü-Ko-Ot 01] were able to prove triviality of Young measures, and hence compactness for the Aviles-Giga functional. Interestingly, the same compactness result was proved independently in [Am-De-Ma 99] by using only the two special entropies introduced by Jin and Kohn [Ji-Ko 00]. This is however not surprising from our new perspective outlined above. Indeed, the matrix space formed by the two special entropies used in [Ji-Ko 00, Am-De-Ma 99] exhibits a locally convex structure as shown in the work of the authors of the current paper [Lo-Pe 16], and we believe this is the underlying reason why the same compactness result can be obtained using only the two special entropies. We plan to pursue this line of research in future works.

2. Proof Sketch

In Subsection 1.1.2 we sketched how the direct proof of Theorem 5 follows from our general approach. In this section we will sketch briefly the main ideas of the proofs of Theorems 1 and 4.

2.1. **Sketch of proof of Theorem 1.** To illustrate the key ideas, we sketch the proof in the special case where $\deg(f_k) = 2$ for $k = 1, 2, \ldots, M_1$ and $\varpi = 0$. Let $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(0)$. By definition, we have that

$$\int f_k(z) d\mu(z) = f_k\left(\int z\, d\mu(z)\right) = 0 \text{ for } k = 1, 2, \ldots, M_1. \tag{24}$$

If the condition (7) is satisfied, then we can find some $\beta_1, \beta_2, \ldots, \beta_{M_1}$ such that $g(z) := \sum_{k=1}^{M_1} \beta_k f_k \geq 0$ on $\mathbb{R}^n$. By (24), we have $\int g(z) d\mu(z) = 0$, and therefore $\mathrm{Spt}\mu \subset V := \{z : g(z) = 0\}$. Since g is a nonnegative quadratic function, the set V is a subspace. By assumption, we can find another linear combination that is nontrivial and nonnegative on V. This way we can iteratively reduce the support of μ onto subspaces of smaller and smaller dimensions until the support is reduced to the origin.

The necessity part of the proof is a bit more intricate. Suppose there exists a subspace V such that $\sum_{k=1}^{M_1} \beta_k f_k$ is nontrivial and changes sign on V for every $\beta \in S^{M_1-1}$. To construct a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(0)$ it suffices to find a finite collection of Dirac measures $\delta_{\tilde{\zeta}_1}, \delta_{\tilde{\zeta}_2}, \ldots, \delta_{\tilde{\zeta}_{2m}}$ and weights $\gamma_1, \gamma_2, \ldots, \gamma_{2m} \geq 0$ such that

$$\mu := \sum_{k=1}^{2m} \gamma_k \delta_{\tilde{\zeta}_k} \in \mathbb{M}^{pc}_{\mathcal{F}}(0) \text{ and } \sum_{k=1}^{2m} \gamma_k = 1. \tag{25}$$

Now we claim that solving (25) is equivalent to finding points $\zeta_1, \zeta_2, \ldots, \zeta_m$ and weights $\gamma_1, \gamma_2, \ldots, \gamma_m \geq 0$ satisfying the system

$$\begin{pmatrix} f_1(\zeta_1) & f_1(\zeta_2) & \ldots & f_1(\zeta_m) \\ f_2(\zeta_1) & f_2(\zeta_2) & \ldots & f_2(\zeta_m) \\ \ldots & \ldots & & \\ f_{M_1}(\zeta_1) & f_{M_1}(\zeta_2) & \ldots & f_{M_1}(\zeta_m) \\ 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \ldots \\ \ldots \\ \gamma_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ \frac{1}{2} \end{pmatrix}. \tag{26}$$

This is because if we find such points $\zeta_1, \zeta_2, \ldots, \zeta_m$ and solve the above system, then simply defining $\tilde{\zeta}_i = \zeta_i$ for $i = 1, 2, \ldots, m$ and $\tilde{\zeta}_{m+i} = -\zeta_i$ for $i = 1, 2, \ldots, m$ and defining $\gamma_{m+i} = \gamma_i$ for $i = 1, 2, \ldots, m$, we have that $\sum_{i=1}^{2m} \gamma_i \tilde{\zeta}_i = 0$. So $\int z\, d\mu(z) = 0$. Further because the functions $\{f_k\}$ are quadratic, we have $f_k(\tilde{\zeta}_i) = f_k(\tilde{\zeta}_{i+m})$ for $i = 1, 2, \ldots m$ and so (26) implies $\sum_{k=1}^{2m} \gamma_k f_i(\tilde{\zeta}_k) = 0$ for $i = 1, 2, \ldots, M_1$ and hence $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(0)$.

Now letting a_i denote the columns of the matrix in (26), the existence of some $\gamma \in \mathbb{R}^m_+$ such that (26) holds true is equivalent to the statement that the vector $b := \frac{1}{2} e_{M_1+1}$ is in the cone $\mathcal{C} := \{\sum_{i=1}^m \lambda_i a_i : \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}_+\}$. This in turn is equivalent to the condition that b cannot be separated from $\mathcal{C}$ with a hyperplane. Formally this is the statement that

$$\text{if } y \in \mathbb{R}^{M_1+1} \text{ satisfies } y \cdot a_i \geq 0 \text{ for all } i = 1, 2, \ldots, m, \text{ then } y \cdot b \geq 0.$$

This is the content of the Farkas-Minkowski Lemma. Now since by hypothesis $\sum_{k=1}^{M_1} y_k f_k$ changes sign on the subspace V for all nontrivial $y \in \mathbb{R}^{M_1}$, we are able to find a finite collection of points $\{\zeta_j\} \subset V$ that is sufficiently dense in V such that for all nontrivial $y \in \mathbb{R}^{M_1}$ we have that $\sum_{k=1}^{M_1} y_k f_k$ changes sign on $\{\zeta_j\}$. Given a nontrivial $y \in \mathbb{R}^{M_1+1}$ such that $y \cdot a_i \geq 0$ for all i, there exists some j_0 such that $\sum_{k=1}^{M_1} y_k f_k(\zeta_{j_0}) < 0$. However as $a_{j_0} \cdot y = \sum_{k=1}^{M_1} y_k f_k(\zeta_{j_0}) + y_{M_1+1} \geq 0$, we must have that $y_{M_1+1} = y \cdot e_{M_1+1} > 0$ and hence $b \in \mathcal{C}$. Thus the existence of $\gamma \in \mathbb{R}^m_+$ satisfying (26) is established, and we indeed have a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(0)$.

For the general case where we allow functions of degree 3 and arbitrary ϖ, we define an auxiliary function $\tilde{f}$ as in (27), which essentially plays the role of translating the barycenter ϖ

to the origin. The sufficiency part works the same way as sketched above by use of the function $\tilde{f}$. However, for the construction of nontrivial measures in $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$, we cannot simply position points symmetrically around ϖ in order to create a measure whose barycenter is ϖ. Instead we need to incorporate the equation $\sum_{i=1}^{m} \gamma_i \zeta_i = \varpi$ into the linear system (26). This is what is done in equation (52) of Lemma 11, but essentially the main ideas are as sketched above.

2.2. **Sketch of proof of Theorem 4.** The case where $\alpha'(\tilde{\alpha}_2) < 0$ follows easily from a well known result of Šverák . The case where $\alpha'(\tilde{\alpha}_2) > 0$ is the one that requires real work. As in Subsection 2.1, to streamline the sketch, we consider the special case where $\tilde{\alpha} = 0$ and $a(\tilde{\alpha}_2) = 0$. Given s_0, t_0 sufficiently small, let

$$\zeta_0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; \zeta_1 := \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix}, \; \zeta_2 := \begin{pmatrix} -s_0 & 0 \\ 0 & -s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix}$$

and

$$\zeta_3 := \begin{pmatrix} 0 & t_0 \\ a(t_0) & 0 \\ 0 & F(t_0) \end{pmatrix}, \; \zeta_4 := \begin{pmatrix} 0 & -t_0 \\ a(-t_0) & 0 \\ 0 & F(-t_0) \end{pmatrix}.$$

So $\zeta_0, \zeta_1, \ldots, \zeta_4 \in \mathcal{K}_1$. For $0 < \epsilon < 1$ sufficiently small, we construct nontrivial measures supported at the above five points, with weight $1 - \epsilon$ at ζ_0, and total weight ϵ at the other four points. Let D_1, D_2, D_3 denote the $(1,2), (2,3), (1,3)$ minors of a 3×2 matrix, respectively. We set the matrix

$$A := \begin{pmatrix} D_1(\zeta_1) & D_1(\zeta_2) & D_1(\zeta_3) & D_1(\zeta_4) \\ D_2(\zeta_1) & D_2(\zeta_2) & D_2(\zeta_3) & D_2(\zeta_4) \\ D_3(\zeta_1) & D_3(\zeta_2) & D_3(\zeta_3) & D_3(\zeta_4) \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} s_0^2 & s_0^2 & -t_0 a(t_0) & t_0 a(-t_0) \\ 0 & 0 & a(t_0)F(t_0) & a(-t_0)F(-t_0) \\ \frac{1}{2}s_0^3 & -\frac{1}{2}s_0^3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

By a careful analysis using the special structure of the points ζ_j, we have that $\sum_{i=1}^{3} y_i D_i$ changes sign on $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ for all nontrivial $y \in \mathbb{R}^3$. By virtue of the arguments used for the system (26), if we define $L^{\epsilon}(\gamma) := A\gamma - (0,0,0,\epsilon)^T$, then the Farkas-Minkowski Lemma guarantees existence of $\gamma_0 \in \mathbb{R}^4_+$ such that $L^{\epsilon}(\gamma_0) = 0$. However what we need to solve for a measure in $\mathcal{M}^{pc}(\mathcal{K}_1)$ is $G^{\epsilon}(\gamma) := L^{\epsilon}(\gamma) - Q(\gamma) = 0$, where

$$Q(\gamma) := \begin{pmatrix} D_1\left(\sum_{j=1}^{4} \gamma_j \zeta_j\right) \\ D_2\left(\sum_{j=1}^{4} \gamma_j \zeta_j\right) \\ D_3\left(\sum_{j=1}^{4} \gamma_j \zeta_j\right) \\ 0 \end{pmatrix}.$$

Since G^{ϵ} is a quadratic perturbation of an invertible function, it should seem reasonable that for small enough ϵ, $G^{\epsilon}(\gamma) = 0$ will have a solution. But to actually establish that the solution is nonnegative we carry out an iterative argument inspired by the proof of the inverse function theorem. To this end, we start from the nonnegative solution γ_0 of the linear part $L^{\epsilon}(\gamma) = 0$, and use an iterative argument to solve for γ_k in each step $k > 0$ such that γ_k converges to the actual solution of $G^{\epsilon}(\gamma) = 0$. The convergence of this scheme is guaranteed by choosing ϵ sufficiently small. These are the contents of Lemmas 19 and 20.

What is slightly surprising is that to prove the general case we need to work instead with the set $\mathcal{K}_1^{\alpha}$ defined by (73) of Section 6. This set is essentially a stripping away of the quadratic part of $\mathcal{K}_1$ around a point α. An analogous set $\mathcal{K}_2^{\alpha}$ was introduced by DiPerna [DP 85] in his proof of Theorem 5. It turns out that Null Lagrangian Measures on $\mathcal{K}_i$ can be transformed into Null Lagrangian Measures on $\mathcal{K}_i^\alpha$ and vice versa, and this is essentially the contents of Section 6. In some sense, the sets $\mathcal{K}_i^\alpha$ play the role of simplifying the problem by allowing the assumptions $\tilde{\alpha} = 0$ and $a(\tilde{\alpha}_2) = 0$. Working with $\mathcal{K}_i^\alpha$ turns out to significantly simplify many technical arguments. For this reason we establish the general result by first establishing it for $\mathcal{K}_1^\alpha$ in Section 7 and the argument follows the skeleton outlined above. We will also utilize the set $\mathcal{K}_2^\alpha$ in the proof of Theorem 6 in Section 8.

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3. Notations and preliminary lemmas for Theorem 1

In this section, we gather some notations and preliminary results that will be used in the proof of Theorem 1. Given a subset K of $\mathbb{R}^n$ or $M^{m\times n}$, let $\mathcal{P}(K)$ denote the space of probability measures supported on K. For a fixed point $\varpi \in \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ and a subspace $V \subset \mathbb{R}^n$, we define

$$\tilde{f}(z) := f(z) - \nabla f(\varpi) \cdot (z - \varpi) - f(\varpi) \tag{27}$$

and

$$\widetilde{V} := V + \varpi. \tag{28}$$

Definition 7. *Let $\mathcal{F}$ be a finite collection of homogeneous polynomials on $\mathbb{R}^n$ and V be a subspace in $\mathbb{R}^n$. We define $\mathcal{F}_V$ to be a subset of $\mathcal{F}$ such that $\left\{ f_{\lfloor V} : f \in \mathcal{F}_V \right\}$ forms a basis of the set $\left\{ f_{\lfloor V} : f \in \mathcal{F} \right\}$.*

Lemma 8. *Let V be a subspace in $\mathbb{R}^n$. Then we have*

$$\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi),\ \mathrm{Spt}\mu \subset V \iff \mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi),\ \mathrm{Spt}\mu \subset V.$$

Proof. First note that as $\mathcal{F}_V \subset \mathcal{F}$ it is immediate that $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi) \subset \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$.

Let $\left\{ f_{k_1}, f_{k_2}, \ldots, f_{k_{N_1}} \right\} = \mathcal{F}_V$. Suppose $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ and $\mathrm{Spt}\mu \subset V$. For any $f \in \mathcal{F}$ we have that

$$f_{\lfloor V} = \sum_{i=1}^{N_1} \beta_i f_{k_i \lfloor V}, \text{ where } \beta_i \in \mathbb{R}.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)d\mu(x) &= \int_V f(x)d\mu(x) \\ &= \sum_{i=1}^{N_1} \beta_i \int_V f_{k_i}(x)d\mu(x) \\ &= \sum_{i=1}^{N_1} \beta_i f_{k_i}(\varpi) = f(\varpi), \end{aligned}$$

and hence $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. □

Lemma 9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree 2, then*

$$\tilde{f}(z) = f(z - \varpi). \tag{29}$$

Proof. Recall that $\tilde{f}(z) \overset{(27)}{=} f(z) - \nabla f(\varpi)\cdot(z-\varpi) - f(\varpi)$. Let us define $g(z) := f(z-\varpi)$. First note that $\tilde{f}(\varpi) = g(\varpi) = 0$. Further, we have

$$\nabla \tilde{f}(z) = \nabla f(z) - \nabla f(\varpi) \quad \text{and} \quad \nabla g(z) = \nabla f(z-\varpi).$$

Since f is quadratic, we know ∇f is linear. Therefore, $\nabla g(z) = \nabla f(z) - \nabla f(\varpi) = \nabla \tilde{f}(z)$. This together with $\tilde{f}(\varpi) = g(\varpi) = 0$ implies $\tilde{f} \equiv g$, which is (29). □

Lemma 10. *Let $\mathcal{F} = \{f_1, f_2, \ldots, f_{M_1}\}$ be a set of homogeneous polynomials on $\mathbb{R}^n$ with the property that*

$$2 \leq \deg(f_k) \leq 3 \text{ for } k = 1,2,\ldots,M_1.$$

Let $V \subset \mathbb{R}^n$ be a subspace such that $\mathcal{F}_V = \left\{f_{k_1}, f_{k_2}, \ldots, f_{k_{N_1}}\right\}$ is nonempty, where we have ordered $\mathcal{F}_V$ so that $f_{k_1}, f_{k_2}, \ldots, f_{k_{N_0}}$ have degree 2 and $f_{k_{N_0+1}}, f_{k_{N_0+2}}, \ldots, f_{k_{N_1}}$ have degree 3. Suppose

$$\sum_{i=1}^{N_0} \beta_i f_{k_i}(z) \text{ changes sign for } z \in S^{n-1}\cap V \text{ for every } \beta \in S^{N_0-1} \tag{30}$$

or

$$\deg(f_{k_i}) = 3 \text{ for all } i = 1,2,\ldots,N_1. \tag{31}$$

Then for any $\varpi \in \mathbb{R}^n$, there exist constants $0 < \Theta_0 < \Theta_1 < \infty$ such that for every $\beta \in S^{N_1+n-1}$, we have that

$$\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right)\cdot(z-\varpi) \text{ changes sign for } z \in A(\varpi,\Theta_0,\Theta_1)\cap \widetilde{V}, \tag{32}$$

where $A(\varpi,\Theta_0,\Theta_1) = \{z \in \mathbb{R}^n : \Theta_0 < |z-\varpi| < \Theta_1\}$.

Proof. We have two cases to consider. The first case is where $N_0 = N_1$, and the second case is where $N_0 < N_1$.

Step 1. We establish (32) in the first case, i.e., when $N_0 = N_1$.
Proof of Step 1. Note that by the hypothesis (30), for each $\beta \in S^{N_0-1}$,

$$m_\beta := \inf\left\{\sum_{i=1}^{N_0} \beta_i f_{k_i}(z) : z \in S^{n-1}\cap V\right\} < 0$$

and

$$M_\beta := \sup\left\{\sum_{i=1}^{N_0} \beta_i f_{k_i}(z) : z \in S^{n-1}\cap V\right\} > 0.$$

We claim that

$$Z_p := \inf_{\beta\in S^{N_0-1}} M_\beta > 0 \quad \text{and} \quad Z_n := \sup_{\beta\in S^{N_0-1}} m_\beta < 0. \tag{33}$$

Suppose the left hand side of (33) is false, so we have a subsequence $\{\beta^m\} \subset S^{N_0-1}$ such that $M_{\beta^m} \to 0$. Passing to a subsequence (not relabeled) we have that $\beta^m \to \bar{\beta}$ and $M_{\bar{\beta}} = 0$. Hence $\sum_{i=1}^{N_0} \bar{\beta}_i f_{k_i} \leq 0$ on $S^{n-1}\cap V$, which contradicts (30). This establishes the left hand side of (33). The right hand side can be established in the same way.

Let $\beta \in S^{N_0+n-1}$. We have two subcases to consider, namely, $\sqrt{\sum_{i=1}^{N_0}\beta_i^2} \geq \frac{1}{2}$ or $\sqrt{\sum_{i=1}^{N_0}\beta_i^2} < \frac{1}{2}$. First, assume $\sqrt{\sum_{i=1}^{N_0}\beta_i^2} \geq \frac{1}{2}$ and define $\tilde{\beta} := \frac{(\beta_1,\beta_2,\ldots,\beta_{N_0})}{\sqrt{\sum_{i=1}^{N_0}\beta_i^2}} \in S^{N_0-1}$. We can find $v_1, v_2 \in$

$S^{n-1} \cap V$ such that

$$\sum_{i=1}^{N_0} \tilde{\beta}_i f_{k_i}(v_1) = m_{\tilde{\beta}} \quad \text{and} \quad \sum_{i=1}^{N_0} \tilde{\beta}_i f_{k_i}(v_2) = M_{\tilde{\beta}}. \tag{34}$$

Note that $f_{k_i}, i = 1, 2, \dots, N_0$, are all homogeneous polynomials of degree 2. For all $\lambda \in \mathbb{R}$, it follows from (29), (34) and (33) that

$$\begin{aligned} &\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(\lambda v_1 + \varpi) + \left(\beta_{N_0+1}, \beta_{N_0+2}, \dots, \beta_{N_0+n}\right) \cdot (\lambda v_1) \\ &\quad \overset{(29)}{=} \sum_{i=1}^{N_0} \beta_i f_{k_i}(\lambda v_1) + \left(\beta_{N_0+1}, \beta_{N_0+2}, \dots, \beta_{N_0+n}\right) \cdot (\lambda v_1) \\ &\quad \overset{(34)}{=} m_{\tilde{\beta}} \lambda^2 \sqrt{\sum_{i=1}^{N_0} \beta_i^2} + \left(\beta_{N_0+1}, \beta_{N_0+2}, \dots, \beta_{N_0+n}\right) \cdot (v_1)\, \lambda \\ &\quad \leq \frac{m_{\tilde{\beta}}}{2} \lambda^2 + |\lambda| \overset{(33)}{\leq} \frac{Z_n}{2} \lambda^2 + |\lambda|. \end{aligned} \tag{35}$$

In a similar way, we obtain

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(\lambda v_2 + \varpi) + \left(\beta_{N_0+1}, \beta_{N_0+2}, \dots, \beta_{N_0+n}\right) \cdot (\lambda v_2) \geq \frac{Z_p}{2} \lambda^2 - |\lambda|. \tag{36}$$

Note that since $v_j \in V$, we have $\lambda v_j + \varpi \in \tilde{V}$ for $j = 1, 2$. It is clear from (35) and (36) that there exist constants $0 < \Theta_0^a < \Theta_1^a < \infty$ depending only on Z_n and Z_p, and thus independent of β, such that (32) holds true on $A(\varpi, \Theta_0^a, \Theta_1^a) \cap \widetilde{V}$, provided $\sqrt{\sum_{i=1}^{N_0} \beta_i^2} \geq \frac{1}{2}$.

Now we consider the subcase where $\sqrt{\sum_{i=1}^{N_0} \beta_i^2} < \frac{1}{2}$, so letting $\hat{\beta} = \left(\beta_{N_0+1}, \beta_{N_0+2}, \dots, \beta_{N_0+n}\right)$, we have $|\hat{\beta}| \geq \frac{1}{2}$. There exists $c_0 = c_0(f_1, f_2, \dots, f_{M_1}) > 0$ such that for all $\lambda \in (0, c_0)$, we have

$$\begin{aligned} &\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(\lambda \hat{\beta} + \varpi) + \hat{\beta} \cdot (\lambda \hat{\beta}) \overset{(29)}{=} \sum_{i=1}^{N_0} \beta_i f_{k_i}(\lambda \hat{\beta}) + \hat{\beta} \cdot (\lambda \hat{\beta}) \\ &\quad = \lambda^2 \left(\sum_{i=1}^{N_0} \beta_i f_{k_i}(\hat{\beta}) \right) + \lambda \left|\hat{\beta}\right|^2 \\ &\quad \geq -\lambda^2 \sqrt{N_0} \max \left\{ \|f_{k_i}\|_{L^\infty(B_1)} : i = 1, 2, \dots, N_0 \right\} + \lambda \left|\hat{\beta}\right|^2 \geq \frac{\lambda}{8}. \end{aligned} \tag{37}$$

Similarly

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(-\lambda \hat{\beta} + \varpi) + \hat{\beta} \cdot (-\lambda \hat{\beta}) \leq -\frac{\lambda}{8} \tag{38}$$

for all $\lambda \in (0, c_0)$. So from (37) and (38) there exist constants $0 < \Theta_0^b < \Theta_1^b < \infty$ depending only on $\mathcal{F}$ such that (32) holds true in $A(\varpi, \Theta_0^b, \Theta_1^b) \cap \widetilde{V}$.

Now letting $\tilde{\Theta}_0 = \min\left\{\Theta_0^a, \Theta_0^b\right\}$ and $\tilde{\Theta}_1 = \max\left\{\Theta_1^a, \Theta_1^b\right\}$ we have that (32) holds true in the case $N_0 = N_1$.

Step 2. We establish (32) in the second case, i.e., when $N_0 < N_1$.
Proof of Step 2. Given a fixed $\varpi \in \mathbb{R}^n$, let $\beta \in S^{N_1+n-1}$. If

$$\beta_i = 0 \text{ for all } i = N_0 + 1, \dots, N_1, \tag{39}$$

then (32) follows from arguments in Step 1. So we can assume (39) is false. As $N_0 < N_1$, we know $\left\{f_{k_{N_0+1}}, f_{k_{N_0+2}} \dots f_{k_{N_1}}\right\} \neq \emptyset$. Since $\left\{f_{k_{N_0+1}} \lfloor_V, f_{k_{N_0+2}} \lfloor_V, \dots, f_{k_{N_1}} \lfloor_V\right\}$ is a linearly independent set and (39) is false, it follows that $\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i} \lfloor_V \not\equiv 0$. Hence there exists some $v_0 \in V \cap S^{n-1}$ such that

$$\Upsilon = \sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(v_0) \neq 0.$$

We can without loss of generality assume $\Upsilon > 0$. For all $\lambda \in \mathbb{R}$, using (29) of Lemma 9 we have

$$\begin{aligned}\left[\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}\right](\lambda v_0 + \varpi) &\overset{(29),(27)}{=} \sum_{i=1}^{N_0} \beta_i f_{k_i}(\lambda v_0) + \sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\lambda v_0 + \varpi) \\ &\quad - \left(\sum_{i=N_0+1}^{N_1} \beta_i \nabla f_{k_i}(\varpi)\right) \cdot (\lambda v_0) - \left(\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\varpi)\right).\end{aligned} \tag{40}$$

Since all the terms are quadratic or linear in λ or constants, there exists some constant $\mathcal{C}_1 > 0$ depending only on $|\beta|$, $|v_0|$, $|\varpi|$ and the polynomials f_{k_i} such that

$$\begin{aligned}&\left|\sum_{i=1}^{N_0} \beta_i f_{k_i}(\lambda v_0)\right| + \left|\left(\sum_{i=N_0+1}^{N_1} \beta_i \nabla f_{k_i}(\varpi)\right) \cdot (\lambda v_0)\right| \\ &+ \left|\left(\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\varpi)\right)\right| + \left|\left(\beta_{N_1+1}, \beta_{N_1+2}, \dots, \beta_{N_1+n}\right) \cdot (\lambda v_0)\right| \leq \mathcal{C}_1 \lambda^2 \text{ for all } |\lambda| \geq 1.\end{aligned} \tag{41}$$

Indeed if we look closer, since $|\beta| = 1$ and $|v_0| = 1$, we see that the above constant $\mathcal{C}_1$ can be made independent of β and v_0. We also know that there exists some constant $\mathcal{C}_2 > 0$ independent of β such that

$$\left|D\left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(z)\right]\right| \leq \mathcal{C}_2 |z|^2 \text{ for all } z \in \mathbb{R}^n.$$

So for all $\lambda \in \mathbb{R}$ with $|\lambda|$ sufficiently large, we have

$$\left|\left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\varpi + \lambda v_0)\right] - \left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\lambda v_0)\right]\right| \leq 4|\varpi| \mathcal{C}_2 |\lambda|^2. \tag{42}$$

Clearly there exists $L > 1$ such that $\frac{\Upsilon}{2}\lambda^3 > (\mathcal{C}_1 + 4|\varpi|\mathcal{C}_2)\lambda^2$ for all $\lambda > L$. It follows from this, (40), (41) and (42) that, for all $\lambda > L$, we have

$$\begin{aligned}&\left[\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}\right](\lambda v_0 + \varpi) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \dots, \beta_{N_1+n}\right) \cdot (\lambda v_0) \\ &\overset{(40),(41),(42)}{\geq} \lambda^3 \sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(v_0) - (\mathcal{C}_1 + 4|\varpi|\mathcal{C}_2)\lambda^2 \geq \frac{\Upsilon}{2}\lambda^3.\end{aligned} \tag{43}$$

In the same way, for all $\lambda < -L$ we have

$$\left[\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}\right](\lambda v_0 + \varpi) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \dots, \beta_{N_1+n}\right) \cdot (\lambda v_0) \leq \frac{\Upsilon}{2}\lambda^3. \tag{44}$$

We deduce from (43)-(44) that there exist constants $0 < \tilde{\tilde{\Theta}}_0 < L < \tilde{\tilde{\Theta}}_1 < \infty$ such that

$$\inf\left\{\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1},\dots,\beta_{N_1+n}\right)\cdot(z-\varpi) : z \in A(\varpi,\tilde{\tilde{\Theta}}_0,\tilde{\tilde{\Theta}}_1)\cap \widetilde{V}\right\} < -L^3\frac{\Upsilon}{2}$$

and

$$\sup\left\{\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1},\dots,\beta_{N_1+n}\right)\cdot(z-\varpi) : z \in A(\varpi,\tilde{\tilde{\Theta}}_0,\tilde{\tilde{\Theta}}_1)\cap \widetilde{V}\right\} > L^3\frac{\Upsilon}{2}.$$

Finally, define $\Theta_0 := \min\{\tilde{\Theta}_0, \tilde{\tilde{\Theta}}_0\}$ and $\Theta_1 := \max\{\tilde{\Theta}_1, \tilde{\tilde{\Theta}}_1\}$. Combining Steps 1 and 2, we obtain that $\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1},\dots,\beta_{N_1+n}\right)\cdot(z-\varpi)$ changes sign on $A(\varpi,\Theta_0,\Theta_1)\cap\widetilde{V}$. □

Lemma 11. *Under the assumptions of Lemma 10, we can construct a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ for any $\varpi \in \mathbb{R}^n$.*

Proof. By Lemma 10 there exist $\Theta_1 > \Theta_0 > 0$ such that (32) holds true. For each $m \in \mathbb{N}$ there exists a finite collection of points $\Pi_m = \left\{\zeta_j^m : j = 1,2,\dots,P_m\right\} \subset \widetilde{V}\cap A(\varpi, \frac{\Theta_0}{2}, 2\Theta_1)$ such that $\widetilde{V}\cap A(\varpi,\Theta_0,\Theta_1) \subset \bigcup_{j=1}^{P_m} B_{2^{-m}}(\zeta_j^m)$.

Step 1. There exists $m \in \mathbb{N}$ such that for every $\beta \in S^{N_1+n-1}$ we have

$$\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1},\beta_{N_1+2},\dots,\beta_{N_1+n}\right)\cdot(z-\varpi) \text{ changes sign on } \Pi_m. \tag{45}$$

Proof of Step 1. Suppose this is false, so for all m we can find $\beta^m \in S^{N_1+n-1}$ such that

$$\sum_{i=1}^{N_1} \beta_i^m \tilde{f}_{k_i}(z) + \left(\beta^m_{N_1+1},\beta^m_{N_1+2},\dots,\beta^m_{N_1+n}\right)\cdot(z-\varpi) \geq 0 \text{ for } z \in \Pi_m. \tag{46}$$

Now passing to a subsequence we have $\beta^m \to \bar{\beta} \in S^{N_1+n-1}$. Note that there exists a constant $\mathcal{C}_3 > 0$ such that

$$\|\sum_{i=1}^{N_1} \bar{\beta}_i D\tilde{f}_{k_i}\|_{L^\infty(B_{\Theta_1}(\varpi))} \leq \mathcal{C}_3. \tag{47}$$

Now by (32) there exists $\zeta \in A(\varpi,\Theta_0,\Theta_1)$ such that

$$\sum_{i=1}^{N_1} \bar{\beta}_i \tilde{f}_{k_i}(\zeta) + \left(\bar{\beta}_{N_1+1},\bar{\beta}_{N_1+2},\dots,\bar{\beta}_{N_1+n}\right)\cdot(\zeta-\varpi) = \delta < 0.$$

Also by (47) for all large enough m we can find $j_m \in \{1,2,\dots,P_m\}$ such that $\zeta^m_{j_m}$ is close enough to ζ so that

$$\sum_{i=1}^{N_1} \bar{\beta}_i \tilde{f}_{k_i}(\zeta^m_{j_m}) + \left(\bar{\beta}_{N_1+1},\bar{\beta}_{N_1+2},\dots,\bar{\beta}_{N_1+n}\right)\cdot\left(\zeta^m_{j_m}-\varpi\right) < \frac{\delta}{2}.$$

Since $\beta^m \to \bar{\beta}$, this implies

$$\sum_{i=1}^{N_1} \beta_i^m \tilde{f}_{k_i}(\zeta^m_{j_m}) + \left(\beta^m_{N_1+1},\beta^m_{N_1+2},\dots,\beta^m_{N_1+n}\right)\cdot\left(\zeta^m_{j_m}-\varpi\right) < \frac{\delta}{4}$$

for all large enough m, which contradicts (46). This completes the proof of Step 1.

Proof of Lemma 11 completed. Define $\Pi := \left\{\zeta_j^m : j = 1, 2, \ldots, P_m\right\}$. Now we will show that we can find coefficients $\gamma_1, \gamma_2, \ldots, \gamma_{P_m} \in \mathbb{R}_+$ for which

$$\sum_{j=1}^{P_m} \gamma_j = 1 \tag{48}$$

and for measure μ defined by $\mu := \sum_{j=1}^{P_m} \gamma_j \delta_{\zeta_j^m}$ we have $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$.

To establish $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ it suffices to show

$$\int z d\mu(z) = \varpi \tag{49}$$

and

$$\int \tilde{f}_{k_i}(z) d\mu(z) = 0 \text{ for } i = 1, 2, \ldots, N_1, \tag{50}$$

because $\int \tilde{f}_{k_i}(z) d\mu(z) \overset{(27)}{=} \int f_{k_i}(z) d\mu(z) - f_{k_i}(\varpi)$. So to establish (50) for the measure μ we need

$$\sum_{j=1}^{P_m} \gamma_j \tilde{f}_{k_i}(\zeta_j^m) = 0 \text{ for } i = 1, 2, \ldots, N_1. \tag{51}$$

Now (48), (49), (51) are equivalent to the system

$$\begin{pmatrix} \tilde{f}_{k_1}(\zeta_1^m) & \tilde{f}_{k_1}(\zeta_2^m) & \ldots & \tilde{f}_{k_1}(\zeta_{P_m}^m) \\ \tilde{f}_{k_2}(\zeta_1^m) & \tilde{f}_{k_2}(\zeta_2^m) & \ldots & \tilde{f}_{k_2}(\zeta_{P_m}^m) \\ \ldots & & & \\ \ldots & & & \\ \tilde{f}_{k_{N_1}}(\zeta_1^m) & \tilde{f}_{k_{N_1}}(\zeta_2^m) & \ldots & \tilde{f}_{k_{N_1}}(\zeta_{P_m}^m) \\ [\zeta_1^m]_1 & [\zeta_2^m]_1 & \ldots & \left[\zeta_{P_m}^m\right]_1 \\ [\zeta_1^m]_2 & [\zeta_2^m]_2 & \ldots & \left[\zeta_{P_m}^m\right]_2 \\ \ldots & & & \\ [\zeta_1^m]_n & [\zeta_2^m]_n & \ldots & \left[\zeta_{P_m}^m\right]_n \\ 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \gamma_{P_m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ [\varpi]_1 \\ [\varpi]_2 \\ \ldots \\ [\varpi]_n \\ 1 \end{pmatrix}, \tag{52}$$

where for a vector v, we denote by $[v]_p$ the p-th component of v. Let

$$a_j := \begin{pmatrix} \tilde{f}_{k_1}(\zeta_j^m) \\ \tilde{f}_{k_2}(\zeta_j^m) \\ \ldots \\ \ldots \\ \tilde{f}_{k_{N_1}}(\zeta_j^m) \\ \left[\zeta_j^m\right]_1 \\ \left[\zeta_j^m\right]_2 \\ \ldots \\ \left[\zeta_j^m\right]_n \\ 1 \end{pmatrix} \text{ for } j = 1, 2, \ldots, P_m \text{ and } b := \begin{pmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ [\varpi]_1 \\ [\varpi]_2 \\ \ldots \\ [\varpi]_n \\ 1 \end{pmatrix}.$$

By the Farkas-Minkowski Lemma the above system (52) has a nonnegative solution (componentwise) if and only if

$$\text{whenever } y \in \mathbb{R}^{N_1+n+1} \text{ and } y \cdot a_j \geq 0 \text{ for } j = 1, 2, \ldots, P_m \text{ then } y \cdot b \geq 0. \tag{53}$$

The hypothesis of (53) is equivalent to

$$\begin{pmatrix} \sum_{i=1}^{N_1} y_i\tilde{f}_{k_i}(\zeta_1^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\zeta_1^m + y_{N_1+n+1} \\ \sum_{i=1}^{N_1} y_i\tilde{f}_{k_i}(\zeta_2^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\zeta_2^m + y_{N_1+n+1} \\ \dots \\ \dots \\ \sum_{i=1}^{N_1} y_i\tilde{f}_{k_i}(\zeta_{P_m}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\zeta_{P_m}^m + y_{N_1+n+1} \end{pmatrix} = \begin{pmatrix} \geq 0 \\ \geq 0 \\ \dots \\ \dots \\ \geq 0 \end{pmatrix}. \tag{54}$$

So pick $y \in \mathbb{R}^{N_1+n+1}$ such that (54) holds true. By (45) for some $j_0 \in \{1, 2, \dots, P_m\}$ we have that

$$\sum_{i=1}^{N_1} y_i\tilde{f}_{k_i}(\zeta_{j_0}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\left(\zeta_{j_0}^m - \varpi\right) < 0, \tag{55}$$

and from (54) we know

$$\sum_{i=1}^{N_1} y_i\tilde{f}_{k_i}(\zeta_{j_0}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\zeta_{j_0}^m + y_{N_1+n+1} \geq 0. \tag{56}$$

Now applying (55) to (56) we have that

$$b\cdot y = \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right)\cdot\varpi + y_{N_1+n+1} > 0.$$

Thus (53) is satisfied and so we can solve system (52) for a nonnegative solution γ. Hence we have a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$. □

4. Proof of Theorem 1 and Corollary 2

In this section, we first give the proof of Theorem 1 using the preliminary results given in the previous section.

Proof of Theorem 1. Let $\mathcal{A} := \text{Span}\left\{f_1, f_2, \dots, f_{M_0}\right\}$. Suppose for every subspace $V \subset \mathbb{R}^n$ the condition (7) holds true. Let $\varpi \in \mathbb{R}^n$ and $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. Now by hypothesis there exists a function $g_1 \in \mathcal{A}$ such that $g_1 \geq 0$ on $\mathbb{R}^n$. Let $V_1 := \{z \in \mathbb{R}^n : g_1(z) = 0\}$. Since g_1 is a quadratic and nonnegative function, it is convex, and hence V_1 is a subspace of $\mathbb{R}^n$. Note that $\widetilde{V}_1 \overset{(29)}{=} \{z \in \mathbb{R}^n : \tilde{g}_1(z) = 0\}$. From (6) and (29) we have $\int \tilde{g}_1(z)d\mu = 0$, and therefore $\text{Spt}\mu \subset \widetilde{V}_1$. Now by hypothesis there exists $g_2 \in \mathcal{A}$ that is nonnegative and nontrivial on V_1. Let $V_2 := \{z \in V_1 : g_2(z) = 0\}$. Note that V_2 is a proper subspace of V_1, and thus $\dim V_2 \leq \dim V_1 - 1$. By applying (29) of Lemma 9 we have

$$\begin{aligned} \widetilde{V}_2 &\overset{(27)}{=} V_2 + \varpi \\ &= \{z + \varpi : z \in V_1, g_2(z) = 0\} \\ &\overset{(28)}{=} \left\{\zeta : \zeta \in \widetilde{V}_1, g_2(\zeta - \varpi) = 0\right\} \\ &\overset{(29)}{=} \left\{\zeta : \zeta \in \widetilde{V}_1, \tilde{g}_2(\zeta) = 0\right\}. \end{aligned} \tag{57}$$

As $\int \tilde{g}_2(z)d\mu = 0$, we have $\text{Spt}\mu \subset \widetilde{V}_2$. Repeating this process, we can find a sequence of subspaces $\mathbb{R}^n \supset V_1 \supset V_2 \supset \cdots \supset V_Q$ with strictly decreasing dimensions and V_Q a one dimensional subspace. Also we have $\text{Spt}\mu \subset \widetilde{V}_Q$. Now there exists a function $g_{Q+1} \in \mathcal{A}$ such that $g_{Q+1} \geq 0$ on V_Q and g_{Q+1} is nontrivial on V_Q. Thus g_{Q+1} is convex and quadratic on V_Q. It follows that $\{0\} = \left\{z \in V_Q : g_{Q+1}(z) = 0\right\}$, and hence (as in (57)) by Lemma 9, we have $\{\varpi\} = \left\{z \in \widetilde{V}_Q : \widetilde{g}_{Q+1}(z) = 0\right\}$. Now as $\int \widetilde{g}_{Q+1}(z)d\mu = 0$, $\text{Spt}\mu = \{\varpi\}$ and hence μ is a Dirac measure.

Conversely suppose for some subspace V, condition (7) fails to holds. So

$$\text{for all } g \in \mathrm{Span}(\{f_1, f_2, \dots, f_{M_0}\}), g \text{ changes sign on } V \text{ or is trivial.} \tag{58}$$

We first consider the case when

$$f_{k\lfloor V} \equiv 0 \text{ for } k = 1, 2, \dots, M_1. \tag{59}$$

Let $q = \dim(V)$. Then for any $\varpi \in V$, let $\nu := \pi^{-1}\mathcal{H}^q_{\lfloor B_1(\varpi)\cap V}$. Note that

$$\begin{aligned}\int f_k(z)d\nu(z) = 0 &= f_k(\varpi)\\ &= f_k\left(\int z d\nu(z)\right) \text{ for all } k = 1, 2, \dots, M_1\end{aligned}$$

and $\bar{\nu} = \varpi$. So $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$.

Now suppose (59) does not hold true, then $\mathcal{F}_V$ is a nonempty subset of $\mathcal{F}$. Let $\mathcal{F}_V = \left\{f_{k_1}, f_{k_2}, \dots, f_{k_{N_1}}\right\}$ where we have ordered them so that $\deg(f_{k_i}) = 2$ for $i = 1, 2, \dots, N_0$ and $\deg(f_{k_i}) = 3$ for $i = N_0 + 1, N_0 + 2, \dots, N_1$. It is clear that (58) implies that

$$\sum_{i=1}^{N_0} \beta_i f_{k_i} \text{ changes sign or is trivial on } V \text{ for any } \beta \in S^{N_0-1}.$$

Since $\left\{f_{k_i\lfloor V} : i = 1, 2, \dots, N_0\right\}$ is a linearly independent set, if $N_0 > 0$ and $\sum_{i=1}^{N_0} \beta_i f_{k_i\lfloor V} \equiv 0$ then $\beta_1 = \beta_2 = \dots \beta_{N_0} = 0$. Thus (30) of Lemma 10 holds true in this case. On the other hand if $\left\{f_{k_i\lfloor V} : i = 1, 2, \dots, N_0\right\}$ is an empty set, then since (59) is false we must have that (31) of Lemma 10 holds true. By Lemma 11 we can construct a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ for any $\varpi \in \mathbb{R}^n$. Hence by Lemma 8 this gives a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. □

Now we use Theorem 1 to give the

Proof of Corollary 2. Let $M = \dim(K)$. Then there exists a linear isomorphism $\sigma : \mathbb{R}^M \to K$. Let $M_1, M_2, \dots, M_{q_1}$ be all nontrivial minors of the matrices in the subspace K and define $f_k(z) := M_k(\sigma(z))$. Let $\mathcal{F} = \{f_1, f_2, \dots, f_{q_1}\}$.

Step 1. Let $\mu \in \mathcal{P}(K)$ and $\nu \in \mathcal{P}(\mathbb{R}^M)$ be such that $\mu = \sigma_\sharp\nu$, i.e., μ is the push forward of ν under the mapping σ, then $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}$ if and only if $\mu \in \mathcal{M}^{pc}(K)$.

Proof of Step 1. By change of variable formula for push forward measures (see [Am-Fu-Pa 00], P. 32) we have

$$\int f_k(z)d\nu(z) = \int M_k(\sigma(z))d\nu(z) = \int M_k(X)d\mu(X)$$

and

$$M_k\left(\int X d\mu(X)\right) = M_k\left(\int \sigma(z)d\nu(z)\right) = M_k\left(\sigma\left(\int z d\nu(z)\right)\right) = f_k(\bar{\nu}).$$

Putting the above together it is clear that $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}$ if and only if $\mu \in \mathcal{M}^{pc}(K)$.

Proof of Corollary 2 completed. Suppose (8) holds true. Let $V \subset \mathbb{R}^M$ be a subspace and define $L = \sigma(V)$. By hypothesis there exists $\beta \in S^{q_0-1}$ such that

$$\sum_{k=1}^{q_0} \beta_k f_k(z) = \sum_{k=1}^{q_0} \beta_k M_k(\sigma(z)) \geq 0 \text{ for all } z \in V \text{ and } \sum_{k=1}^{q_0} \beta_k f_k = \sum_{k=1}^{q_0} \beta_k (M_k \circ \sigma) \not\equiv 0.$$

So condition (7) holds true. For all $\mu \in \mathcal{M}^{pc}(K)$, it follows from Step 1 that $\nu := (\sigma^{-1})_\sharp \mu \in \mathbb{M}^{pc}_{\mathcal{F}}$. It is not hard to see that the proof of the sufficiency part of Theorem 1 does not rely on the assumption that the degrees of the polynomials are less than or equal to three. Hence, by the sufficiency part of Theorem 1, we have $\nu = \delta_{\overline{\nu}}$ and thus μ is also a Dirac measure.

Conversely suppose condition (8) is false and $2 \leq \min\{m,n\} \leq 3$. Note that all the functions f_k are either homogeneous of degree 2 or 3 (when $\min\{m,n\} = 2$, all f_k are simply homogeneous of degree 2). We assume that we have ordered $\{M_k : k = 1, 2 \ldots, q_1\}$ such that $f_1, f_2, \ldots, f_{q_0}$ are quadratics and $f_{q_0+1}, f_{q_0+2}, \ldots, f_{q_1}$ are cubics. There exists some subspace $L \subset K$ such that $\sum_{k=1}^{q_0} \beta_k M_k$ changes sign or is trivial on L for any $\beta \in S^{q_0-1}$. Define $V := \sigma^{-1}(L)$, then $\sum_{k=1}^{q_0} \beta_k f_k$ changes sign or is trivial on V for any $\beta \in S^{q_0-1}$. Thus, by Theorem 1, for any $\varpi \in V$ we can construct a nontrivial measure $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. Let $\mu := \sigma_\sharp \nu$. By Step 1 we have that $\mu \in \mathcal{M}^{pc}(K)$. As μ is the push forward by a linear isomorphism of a nontrivial measure, it is clear that μ is nontrivial. $\square$

5. Condition (8) for two dimensional subspaces in $M^{m\times n}$ and proof of Corollary 3

In this section, we first show that for two dimensional subspaces in $M^{m\times n}$, the condition of having no Rank-1 connections implies condition (8). As a consequence, we provide the proof of Corollary 3.

5.1. **Condition (8) for two dimensional subspaces.**

Lemma 12. *Let $M_1, M_2, \ldots, M_{q_0}$ denote all 2×2 minors of $M^{m\times n}$. Suppose $K \subset M^{m\times n}$ is a two dimensional subspace without Rank-1 connections, then the following condition holds true:*

$$\text{For each subspace } L \subset K \text{ there exists } \beta \in S^{q_0-1} \text{ such that}$$
$$\sum_{k=1}^{q_0} \beta_k M_k(X) \geq 0 \text{ for all } X \in L \text{ and } \sum_{k=1}^{q_0} \beta_k M_k \not\equiv 0 \text{ on } L. \tag{60}$$

The proof of the above lemma requires some preparation. We begin by introducing some notations. Given $A \in M^{m\times n}$, we denote

$$M^{n_1,n_2}_{m_1,m_2}(A) := \det \begin{pmatrix} a_{m_1 n_1} & a_{m_1 n_2} \\ a_{m_2 n_1} & a_{m_2 n_2} \end{pmatrix}$$

for $m_1 \neq m_2 \in \{1, 2, \ldots, m\}$, $n_1 \neq n_2 \in \{1, 2, \ldots, n\}$. Let $K \subset M^{m\times n}$ be a two dimensional subspace, then there exist $a_{ij} \in \mathbb{R}^2$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ such that

$$P(z) := \begin{pmatrix} a_{11} \cdot z & a_{12} \cdot z & \ldots & a_{1n} \cdot z \\ a_{21} \cdot z & a_{22} \cdot z & \ldots & a_{2n} \cdot z \\ \ldots & & & \\ \ldots & & & \\ a_{m1} \cdot z & a_{m2} \cdot z & \ldots & a_{mn} \cdot z \end{pmatrix} \tag{61}$$

is a parameterization of K. Thus we have

$$K = \left\{ P(z) : z \in \mathbb{R}^2 \right\}.$$

For any $z \neq 0$, if we perform an elementary row or column operation on the matrix $P(z)$ we obtain a matrix which we will denote by $T(z)$. Now due to the linearity of the inner product, $T(z)$ can be represented by

$$T(z) := \begin{pmatrix} \tilde{a}_{11} \cdot z & \ldots & \tilde{a}_{1n} \cdot z \\ \ldots & & \\ \tilde{a}_{m1} \cdot z & \ldots & \tilde{a}_{mn} \cdot z \end{pmatrix}$$

where $\tilde{a}_{ij} \in \mathbb{R}^2$. We first show the following

Lemma 13. *Let $M_1, M_2, \ldots, M_{q_0}$ denote all 2×2 minors of $M^{m\times n}$. Suppose that $K \subset M^{m\times n}$ is a two dimensional subspace parameterized by (61). For $z \neq 0$, let $T(z) \in M^{m\times n}$ be obtained from $P(z)$ by finitely many elementary row and column operations and define*

$$\tilde{K} := \left\{ T(z) : z \in \mathbb{R}^2 \right\}.$$

Then K has no Rank-1 connections if and only if $\tilde{K}$ has no Rank-1 connections. Further, we have

$$\text{Span}\left\{ M_1(P(z)), \ldots, M_{q_0}(P(z)) \right\} = \text{Span}\left\{ M_1(T(z)), \ldots, M_{q_0}(T(z)) \right\}. \tag{62}$$

Proof. By induction, it suffices to consider the case where $T(z)$ is obtained from $P(z)$ by an elementary row or column operation. We only show the case where $T(z)$ is obtained from $P(z)$ by an elementary row operation, as the proof for column operation is identical.

Note that

$$\text{Rank}(T(z)) = \text{Rank}(P(z)) \text{ for any } z \neq 0. \tag{63}$$

Therefore

$$\begin{aligned} & K \text{ has no Rank-1 connections} \\ \iff & \text{Rank}(P(z)) \geq 2 \text{ for any } z \neq 0 \\ \overset{(63)}{\iff} & \text{Rank}(T(z)) \geq 2 \text{ for any } z \neq 0 \\ \iff & \tilde{K} \text{ has no Rank-1 connections.} \end{aligned}$$

Next, as $T(z)$ is obtained from $P(z)$ by an elementary tow operation, we have that

$$R_i(T(z)) = \sum_{i'=1}^{m} c_{i,i'} R_i'(P(z)), \tag{64}$$

where $R_i(A)$ denotes the i-th row of a matrix A. It follows that

$$\begin{aligned} M^{j_0,j_1}_{i_0,i_1}(T(z)) &= \det \begin{pmatrix} [T(z)]_{i_0,j_0} & [T(z)]_{i_0,j_1} \\ [T(z)]_{i_1,j_0} & [T(z)]_{i_1,j_1} \end{pmatrix} \\ &= \left([T(z)]_{i_0,j_0}, [T(z)]_{i_0,j_1} \right) \wedge \left([T(z)]_{i_1,j_0}, [T(z)]_{i_1,j_1} \right) \\ &= \left(\sum_{i'=1}^{m} c_{i_0,i'} \left([P(z)]_{i',j_0}, [P(z)]_{i',j_1} \right) \right) \wedge \left(\sum_{i''=1}^{m} c_{i_1,i''} \left([P(z)]_{i'',j_0}, [P(z)]_{i'',j_1} \right) \right) \\ &= \sum_{i',i''=1}^{m} c_{i_0,i'} c_{i_1,i''} M^{j_0,j_1}_{i',i''}(P(z)). \end{aligned}$$

This shows that the minors of $T(z)$ are inside the span of the minors of $P(z)$. Conversely, by (64) the rows of $P(z)$ can be represented as linear combinations of the rows of $T(z)$. Therefore, exactly the same argument shows the opposite inclusion in (62). □

Proof of Lemma 12. First note that if $m = n = 2$, the lemma is trivial because in this case there is only one minor. If it fails to be positive semidefinite or negative semidefinite then the subspace K has Rank-1 connections. So we can assume $\max\{m, n\} \geq 3$.

We have two cases to consider. The first case is where $\dim(L) = 2$ and the second case is where $\dim(L) = 1$. The former requires much more work and we consider that case first. Note that in this case $L = K$. Assume that K is parameterized by $P(z)$ given in (61). We first claim that there exists $i_0 \in \{1, \ldots, m\}$ such that

$$\dim\left(\text{Span}\left\{ a_{i_0 1}, \ldots, a_{i_0 n} \right\}\right) > 1 \tag{65}$$

or there exists $j_0 \in \{1, \ldots, n\}$ such that

$$\dim\left(\text{Span}\left\{a_{1j_0}, \ldots, a_{mj_0}\right\}\right) > 1. \tag{66}$$

Indeed, if for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we have that (65) and (66) fail, then there exists $a_{i_1j_1} \neq 0 \in \mathbb{R}^2$ such that $a_{ij} = \lambda_{ij} a_{i_1j_1}$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $\lambda_{ij} \in \mathbb{R}$. Now clearly there exists $z \neq 0 \in \mathbb{R}^2$ such that $z \cdot a_{i_1j_1} = 0$, and therefore we have $z \cdot a_{ij} = 0$ for all i, j. We see from (61) that $P(z) = 0$, which contradicts the fact that K has no Rank-1 connections.

Without loss of generality assume (66), and thus there exists $j_0 \in \{1, 2, \ldots, n\}$ and $i_0, i_1 \in \{1, 2, \ldots, m\}$ such that $a_{i_0j_0}$ and $a_{i_1j_0}$ are linearly independent. By performing row and column operations on $P(z)$ we can transform this matrix to $T(z)$ where $\tilde{a}_{11} = a_{i_0j_0}$ and $\tilde{a}_{21} = a_{i_1j_0}$. Note that $a_{ij} \in \mathbb{R}^2$ for all i, j. Therefore, all a_{ij_0} can be expressed as a linear combination of $a_{i_0j_0}$ and $a_{i_1j_0}$. Thus we can perform row operations to further transform $T(z)$ to the matrix

$$\tilde{T}(z) = \begin{pmatrix} \tilde{a}_{11} \cdot z & \tilde{a}_{12} \cdot z & \ldots & \tilde{a}_{1n} \cdot z \\ \tilde{a}_{21} \cdot z & \tilde{a}_{22} \cdot z & \ldots & \tilde{a}_{2n} \cdot z \\ 0 & \tilde{a}_{32} \cdot z & \ldots & \tilde{a}_{3n} \cdot z \\ \ldots & & & \\ \ldots & & & \\ 0 & \tilde{a}_{m2} \cdot z & \ldots & \tilde{a}_{mn} \cdot z \end{pmatrix}.$$

Now we have two subcases to consider. The first subcase is where $\tilde{a}_{i_0j_0} \neq 0$ for some $i_0 \in \{3, \ldots, m\}$ and $j_0 \in \{2, \ldots, n\}$, and the second subcase is where $\tilde{a}_{ij} = 0$ for all $i \in \{3, \ldots, m\}$ and $j \in \{2, \ldots, n\}$.

Step 1. We consider the first subcase where $\tilde{a}_{i_0j_0} \neq 0$ for some $i_0 \in \{3, \ldots, m\}$ and $j_0 \in \{2, \ldots, n\}$. Now as $\tilde{a}_{11}$ and $\tilde{a}_{21}$ are linearly independent, we may write $\tilde{a}_{i_0j_0} = \tilde{\beta}_1 \tilde{a}_{11} + \tilde{\beta}_1 \tilde{a}_{21}$ for some $\tilde{\beta}_1, \tilde{\beta}_2 \in \mathbb{R}$. It follows that

$$\begin{aligned} &\tilde{\beta}_1 M^{1,j_0}_{1,i_0}\left(\tilde{T}(z)\right) + \tilde{\beta}_2 M^{1,j_0}_{2,i_0}\left(\tilde{T}(z)\right) \\ &\quad = \tilde{\beta}_1 \det\begin{pmatrix} \tilde{a}_{11} \cdot z & \tilde{a}_{1j_0} \cdot z \\ 0 & \tilde{a}_{i_0j_0} \cdot z \end{pmatrix} + \tilde{\beta}_2 \det\begin{pmatrix} \tilde{a}_{21} \cdot z & \tilde{a}_{2j_0} \cdot z \\ 0 & \tilde{a}_{i_0j_0} \cdot z \end{pmatrix} \\ &\quad = \left((\tilde{\beta}_1 \tilde{a}_{11} + \tilde{\beta}_2 \tilde{a}_{21}) \cdot z\right)\left(\tilde{a}_{i_0j_0} \cdot z\right) \\ &\quad = \left|\tilde{a}_{i_0j_0} \cdot z\right|^2, \end{aligned} \tag{67}$$

which is nonnegative and nontrivial. Hence we obtain from (62) that there exists a linear combination of the 2×2 minors of $P(z)$ that is nonnegative and nontrivial. This establishes (60) in subcase 1.

Step 2. We consider the second subcase where $\tilde{a}_{ij} = 0$ for all $i \in \{3, \ldots, m\}$ and $j \in \{2, \ldots, n\}$. We know from Lemma 13 that $\text{Rank}\left(\tilde{T}(z)\right) = \text{Rank}\left(P(z)\right) > 1$ for all $z \neq 0 \in \mathbb{R}^2$. Now we claim that

$$\dim\left(\text{Span}\left\{\tilde{a}_{11}, \ldots, \tilde{a}_{1n}\right\}\right) = 2.$$

Suppose not, then for some $\tilde{a}_{1j_0} \neq 0 \in \mathbb{R}^2$, we have $\tilde{a}_{1j} = \lambda_j \tilde{a}_{1j_0}$ for all $j \in \{1, \ldots, n\}$ and $\lambda_j \in \mathbb{R}$. Then we can choose $z \neq 0 \in \mathbb{R}^2$ such that $z \cdot \tilde{a}_{1j_0} = 0$, and thus this z makes the first row of $\tilde{T}(z)$ zero, which is a contradiction to the fact that $\tilde{T}(z)$ has no Rank-1 connections.

Without loss of generality due to Lemma 13, we may assume that $\tilde{a}_{11}$ and $\tilde{a}_{12}$ are linearly independent. Similar to before, we can perform column operations to reduce the matrix $\tilde{T}(z)$

to

$$\tilde{\tilde{T}}(z) = \begin{pmatrix} \tilde{\tilde{a}}_{11}\cdot z & \tilde{\tilde{a}}_{12}\cdot z & 0 & \dots & 0 \\ \tilde{\tilde{a}}_{21}\cdot z & \tilde{\tilde{a}}_{22}\cdot z & \tilde{\tilde{a}}_{23}\cdot z & \dots & \tilde{\tilde{a}}_{2n}\cdot z \\ 0 & 0 & 0 & \dots & 0 \\ \dots & & & & \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

If $\tilde{\tilde{a}}_{2j_0} \neq 0$ for some $j_0 \in \{3,\dots,n\}$, then we can write $\tilde{\tilde{a}}_{2j_0} = \tilde{\tilde{\beta}}_1 \tilde{\tilde{a}}_{11} + \tilde{\tilde{\beta}}_2 \tilde{\tilde{a}}_{12}$ for $\tilde{\tilde{\beta}}_1, \tilde{\tilde{\beta}}_2 \in \mathbb{R}$, and similar to (67), we have

$$\tilde{\tilde{\beta}}_1 M^{1,j_0}_{1,2}\left(\tilde{\tilde{T}}(z)\right) + \tilde{\tilde{\beta}}_2 M^{2,j_0}_{1,2}\left(\tilde{\tilde{T}}(z)\right) = \left|\tilde{\tilde{a}}_{2j_0}\cdot z\right|^2.$$

Hence, by (62), there exists a linear combination of the 2×2 minors of $P(z)$ that is nonnegative and nontrivial.

On the other hand, if $\tilde{\tilde{a}}_{2j} = 0$ for all $j \in \{3,\dots,n\}$, then since $\tilde{\tilde{T}}(z)$ has no Rank-1 connections, it is clear that $\det\begin{pmatrix} \tilde{\tilde{a}}_{11}\cdot z & \tilde{\tilde{a}}_{12}\cdot z \\ \tilde{\tilde{a}}_{21}\cdot z & \tilde{\tilde{a}}_{22}\cdot z \end{pmatrix}$ is either positive semidefinite or negative semidefinite. In either case, (60) follows trivially by (62). This establishes subcase 2.

Proof of Lemma 12 completed. Finally, we consider the case where $\dim(L) = 1$. Assume that L is parametrized by $\mathbb{P}(t)$ given by

$$\mathbb{P}(t) := \begin{pmatrix} b_{11}t & b_{12}t & \dots & b_{1n}t \\ b_{21}t & b_{22}t & \dots & b_{2n}t \\ \dots & & & \\ \dots & & & \\ b_{m1}t & b_{m2}t & \dots & b_{mn}t \end{pmatrix}$$

for $b_{ij} \in \mathbb{R}$. It is not hard to see that Lemma 13 still holds true for one dimensional subspaces. Using elementary row operations, we reduce the matrix

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & & & \\ \dots & & & \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

to the triangle form

$$\begin{pmatrix} 0 & \dots & 0 & \tilde{b}_{1j_1} & \dots & \tilde{b}_{1j_2} & \dots & \tilde{b}_{1m} \\ 0 & \dots & 0 & 0 & \dots & \tilde{b}_{2j_2} & \dots & \tilde{b}_{2m} \\ \dots & & & & & & & \\ \dots & & & & & & & \\ 0 & \dots & 0 & 0 & \dots & * & \dots & * \end{pmatrix}$$

and denote

$$\mathbb{T}(t) := \begin{pmatrix} 0 & \dots & 0 & \tilde{b}_{1j_1}t & \dots & \tilde{b}_{1j_2}t & \dots & \tilde{b}_{1m}t \\ 0 & \dots & 0 & 0 & \dots & \tilde{b}_{2j_2}t & \dots & \tilde{b}_{2m}t \\ \dots & & & & & & & \\ \dots & & & & & & & \\ 0 & \dots & 0 & 0 & \dots & * & \dots & * \end{pmatrix},$$

where $\tilde{b}_{ij_i}$ is the first nonzero entry in the i-th row. Note that the second row in $\mathbb{T}(t)$ must be nontrivial, as otherwise $\mathbb{P}(t)$ would form a Rank-1 line in the subspace K, which is a contradiction. Now

$$\det\begin{pmatrix} \tilde{b}_{1j_1}t & \tilde{b}_{1j_2}t \\ 0 & \tilde{b}_{2j_2}t \end{pmatrix} = \tilde{b}_{1j_1}\tilde{b}_{2j_2}t^2$$

is nontrivial. Hence it follows from Lemma 13 for one dimensional subspaces that (60) holds in the case where $\dim(L) = 1$. This completes the proof of Lemma 12. □

5.2. **Proof of Corollary 3.** Now we give the proof of Corollary 3.

Proof of Corollary 3. By Lemma 12, if K has no Rank-1 connections then (60) of Lemma 12, and hence (8) of Corollary 2, are satisfied. Thus, by Corollary 2, any $\mu \in \mathcal{M}^{pc}(K)$ is a Dirac measure. Now suppose that K has Rank-1 connections, then there exists a nontrivial subspace $L \subset K$ such that $\mathrm{Rank}(A) = 1$ for any $A \in L$. Pick $A_1, A_2 \in L$, $A_1 \neq A_2$, and let $\mu := \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2}$. Note that for any minor M_q of $M^{m\times n}$, $M_q(A_1) = M_q(A_2) = M_q(\frac{A_1+A_2}{2}) = 0$. It follows that

$$\int_K M_q(X)d\mu(X) = \frac{1}{2}M_q(A_1) + \frac{1}{2}M_q(A_2) = 0 = M_q\left(\frac{A_1+A_2}{2}\right) = M_q\left(\int_K X d\mu(X)\right).$$

Hence $\mu \in \mathcal{M}^{pc}(K)$ and $\mathcal{M}^{pc}(K)$ contains nontrivial measures. □

6. Preliminaries for Theorems 4 and 6

In this section, we gather some preliminary lemmas that will be useful in dealing with $\mathcal{M}^{pc}(\mathcal{K}_1)$ and $\mathcal{M}^{pc}(\mathcal{K}_2)$ in the following two sections. First, we introduce some notations that will be used repeatedly. Given a matrix $A \in M^{m\times n}$, let $R_i(A)$ denote the i-th row of A and $[A]_{ij}$ denote the (i,j) element of the matrix A. Let $A \in M^{m\times 2}$ with $m \geq 2$ and $1 \leq i < j \leq m$, we define

$$X_{ij}(A) := \begin{pmatrix} [A]_{i1} & [A]_{i2} \\ [A]_{j1} & [A]_{j2} \end{pmatrix} \tag{68}$$

and

$$M_{ij}(A) := R_i(A) \wedge R_j(A) = \det\left(X_{ij}(A)\right). \tag{69}$$

Given 2×2 matrices A and B, recall that

$$\det(A - B) = \det(A) - A : \mathrm{Cof}(B) + \det(B). \tag{70}$$

Recall the definitions of $\mathcal{K}_1, \mathcal{K}_2, P_1$ and P_2 in (18), (19), (20) and (21), respectively. Further, given $\alpha \in \mathbb{R}^2$, define

$$P_1^\alpha(u,v) := \begin{pmatrix} u-\alpha_1 & v-\alpha_2 \\ a(v)-a(\alpha_2) & u-\alpha_1 \\ (u-\alpha_1)\left(a(v)-a(\alpha_2)\right) & \frac{(u-\alpha_1)^2}{2} + F(v) - F(\alpha_2) - a(\alpha_2)(v-\alpha_2) \end{pmatrix}. \tag{71}$$

and

$$P_2^\alpha(u,v) := \begin{pmatrix} u-\alpha_1 & v-\alpha_2 \\ a(v)-a(\alpha_2) & u-\alpha_1 \\ (u-\alpha_1)\left(a(v)-a(\alpha_2)\right) & \frac{(u-\alpha_1)^2}{2} + F(v) - F(\alpha_2) - a(\alpha_2)(v-\alpha_2) \\ \frac{(u-\alpha_1)^2}{2} + F(\alpha_2) - F(v) + a(v)(v-\alpha_2) & (u-\alpha_1)(v-\alpha_2) \end{pmatrix}. \tag{72}$$

Similar to $\mathcal{K}_1$ and $\mathcal{K}_2$, we denote

$$\mathcal{K}_1^\alpha := \{P_1^\alpha(u,v) : u, v \in \mathbb{R}\} \tag{73}$$

and

$$\mathcal{K}_2^\alpha := \{P_2^\alpha(u,v) : u,v \in \mathbb{R}\}. \tag{74}$$

Finally, given a measure μ and a function f which is integrable with respect to the measure μ, define

$$\overline{f} := \int f(z)d\mu(z).$$

We prove a couple of lemmas that will be essential in the following two sections.

Lemma 14. *For all $\alpha \in \mathbb{R}^2$ and $k \in \{1,2\}$, the push forward map $(P_k^\alpha)_\sharp : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathcal{K}_k^\alpha)$ defined by $\nu \mapsto \mu := (P_k^\alpha)_\sharp \nu$ forms a bijection. Moreover, $\mu \in \mathcal{M}^{pc}(\mathcal{K}_k^\alpha)$ if and only if*

$$\int_{\mathbb{R}^2} M_{ij}(P_k^\alpha(u,v))d\nu = M_{ij}\left(\int_{\mathbb{R}^2} P_k^\alpha(u,v)d\nu\right) \text{ for } i<j \in \{1,\ldots,k+2\}. \tag{75}$$

Further, given $\delta > 0$, if $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_\delta(0)$ then $\mathrm{Spt}\nu \subset B_\delta(\alpha)$, and conversely, if $\mathrm{Spt}\nu \subset B_\delta(\alpha)$ then $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_{C\delta}(0)$ for some constant C depending on the function a, α_2 and δ.

Proof. First note that, since the first row of $P_k^\alpha(u,v)$ is $(u-\alpha_1, v-\alpha_2)$, it is clear that $P_k^\alpha : \mathbb{R}^2 \to \mathcal{K}_k^\alpha$ is a bijection. Therefore it is straight forward to check that $\left((P_k^\alpha)^{-1}\right)_\sharp$ is the inverse map of $(P_k^\alpha)_\sharp$ and hence $(P_k^\alpha)_\sharp$ is a bijection.

Let $\nu \in \mathcal{P}(\mathbb{R}^2)$ and $\mu \in \mathcal{P}(\mathcal{K}_k^\alpha)$ be related by $\mu = (P_k^\alpha)_\sharp \nu$. By change of variable formula for push forward measures, we have

$$\int_{\mathbb{R}^2} M_{ij}(P_k^\alpha(u,v))d\nu = \int_{\mathcal{K}_k^\alpha} M_{ij}(\zeta)d\mu(\zeta)$$

and

$$M_{ij}\left(\int_{\mathbb{R}^2} P_k^\alpha(u,v))d\nu\right) = M_{ij}\left(\int_{\mathcal{K}_k^\alpha} \zeta d\mu(\zeta)\right).$$

It follows that $\mu \in \mathcal{M}^{pc}(K^\alpha)$ if and only if (75) holds.

Next, assume $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_\delta(0)$. Since the first row of $P_k^\alpha(u,v)$ is $(u-\alpha_1, v-\alpha_2)$, it is clear that $\|(u,v)-\alpha\| \leq \|P_k^\alpha(u,v)\|$. Therefore $(P_k^\alpha)^{-1}(\mathcal{K}_k^\alpha \cap B_\delta(0)) \subset B_\delta(\alpha)$. As $\mathrm{Spt}\nu = \left(P_k^\alpha\right)^{-1}\mathrm{Spt}\mu$, it follows that $\mathrm{Spt}\nu \subset B_\delta(\alpha)$. Conversely, assume $\mathrm{Spt}\nu \subset B_\delta(\alpha)$. From the expression of $P_k^\alpha(u,v)$ in (71) or (72), it is clear that the absolute value of each component of $P_k^\alpha(u,v)$ is bounded above by $C\|(u,v)-\alpha\|$ for some constant C depending on the function a, α_2 and δ, provided δ is sufficiently small. Therefore $\|P_k^\alpha(u,v)\| \leq \tilde{C}\|(u,v)-\alpha\|$ and hence $P_k^\alpha(B_\delta(\alpha)) \subset \mathcal{K}_k^\alpha \cap B_{\tilde{C}\delta}(0)$. It follows that $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_{\tilde{C}\delta}(0)$. This completes the proof of the lemma. □

The following lemma is implicitly stated in [Ki-Mü-Sv 03]. We thank S. Müller [Mü 18] for providing us with the elegant proof presented in this section.

Lemma 15 (Kirchheim-Müller-Šverák [Ki-Mü-Sv 03])**.** *Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, for all $\alpha \in \mathbb{R}^2$ and $k \in \{1,2\}$, we have*

$$(P_k)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_k) \Longleftrightarrow (P_k^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_k^\alpha). \tag{76}$$

We show (76) for $k=2$, and the case for $k=1$ is included. We break the proof into several steps. The first lemma is standard.

Lemma 16. *Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, for all $\alpha \in \mathbb{R}^2$, we have*

$$(P_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2^\alpha) \Longleftrightarrow (\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha), \tag{77}$$

where

$$\tilde{P}_2^\alpha(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) - \alpha_1 a(v) - ua(\alpha_2) & \frac{u^2}{2} - u\alpha_1 + F(v) - va(\alpha_2) \\ \frac{u^2}{2} - u\alpha_1 - F(v) + va(v) - \alpha_2 a(v) & uv - \alpha_1 v - u\alpha_2 \end{pmatrix} \tag{78}$$

and

$$\tilde{\mathcal{K}}_2^\alpha := \left\{ \tilde{P}_2^\alpha(u,v) : u,v \in \mathbb{R} \right\}.$$

Proof. Recall the definition of P_2^α given in (72). Direct calculations show that

$$P_2^\alpha(u,v) = \tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha \tag{79}$$

for

$$\tilde{E}^\alpha := \begin{pmatrix} \alpha_1 & \alpha_2 \\ a(\alpha_2) & \alpha_1 \\ -\alpha_1 a(\alpha_2) & -\frac{\alpha_1^2}{2} + F(\alpha_2) - \alpha_2 a(\alpha_2) \\ -\frac{\alpha_1^2}{2} - F(\alpha_2) & -\alpha_1\alpha_2 \end{pmatrix}.$$

Note that $\tilde{E}^\alpha$ is the constant part of $P_2^\alpha(u,v)$.

Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, by arguing exactly as in Lemma 14, we have that

$$(P_2)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2) \iff \int_{\mathbb{R}^2} M_{ij}(P_2(u,v))d\nu = M_{ij}\left(\int_{\mathbb{R}^2} P_2(u,v)d\nu\right) \text{ for } i<j \in \{1,2,3,4\}$$

and

$$(\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha) \iff \int_{\mathbb{R}^2} M_{ij}(\tilde{P}_2^\alpha(u,v))d\nu = M_{ij}\left(\int_{\mathbb{R}^2} \tilde{P}_2^\alpha(u,v)d\nu\right) \text{ for } i<j \in \{1,2,3,4\}. \tag{80}$$

Using the formula (70), the fact that $\tilde{E}^\alpha$ is a constant matrix and the notation $M_{ij}(\cdot) = \det\left(X_{ij}(\cdot)\right)$ (recalling (68)), we have

$$\begin{aligned}
&\int M_{ij}\left(P_2^\alpha(u,v)\right)d\nu \overset{(79)}{=} \int M_{ij}\left(\tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha\right)d\nu \\
&= \int \left[M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) - X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right)\right]d\nu \\
&= \int M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu - \int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right).
\end{aligned} \tag{81}$$

In a similar way using (70) we have that

$$\begin{aligned}
&\det\left(\int X_{ij}\left(P_2^\alpha(u,v)\right)d\nu\right) \\
&= \det\left(\int X_{ij}\left(\tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha\right)d\nu\right) \\
&= \det\left(\int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu\right) - \int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right).
\end{aligned} \tag{82}$$

Putting (81) and (82) together and using Lemma 14 we have that

$$
\begin{aligned}
&(P_2^{\alpha})_{\sharp}\nu \in \mathcal{M}^{pc}(\mathcal{K}_2^{\alpha})\\
&\overset{(75)}{\Longleftrightarrow} \int M_{ij}\left(P_2^{\alpha}(u,v)\right) d\nu = \det\left(\int X_{ij}\left(P_2^{\alpha}(u,v)\right) d\nu\right) \text{ for all } i,j\\
&\overset{(81),(82)}{\Longleftrightarrow} \int M_{ij}\left(\tilde{P}_2^{\alpha}(u,v)\right) d\nu = \det\left(\int X_{ij}\left(\tilde{P}_2^{\alpha}(u,v)\right) d\nu\right) \text{ for all } i,j\\
&\overset{(80)}{\Longleftrightarrow} (\tilde{P}_2^{\alpha})_{\sharp}\nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^{\alpha}).
\end{aligned}
$$

This establishes (77). □

Lemma 17 (Müller [Mü 18]). *Every row $R_i\left(\tilde{P}_2^{\alpha}(u,v)\right)$ of the matrix $\tilde{P}_2^{\alpha}(u,v)$ can be expressed as a linear combination of the rows of $P_2(u,v)$, and conversely every row $R_i\left(P_2(u,v)\right)$ of the matrix $P_2(u,v)$ can be expressed as a linear combination of the rows of $\tilde{P}_2^{\alpha}(u,v)$, and the coefficients depend only on α, but not on (u,v), i.e.,*

$$
R_i\left(\tilde{P}_2^{\alpha}(u,v)\right) = \sum_{i'=1}^{4} c_{ii'}(\alpha)\, R_{i'}\left(P_2(u,v)\right) \quad \textit{for all } (u,v) \in \mathbb{R}^2 \tag{83}
$$

and

$$
R_i\left(P_2(u,v)\right) = \sum_{i'=1}^{4} \tilde{c}_{ii'}(\alpha)\, R_{i'}\left(\tilde{P}_2^{\alpha}(u,v)\right) \quad \textit{for all } (u,v) \in \mathbb{R}^2. \tag{84}
$$

Proof. From the definitions of P_2 and $\tilde{P}_2^{\alpha}$ in (21) and (78), we see that

$$
R_1\left(\tilde{P}_2^{\alpha}(u,v)\right) = R_1\left(P_2(u,v)\right), \quad R_2\left(\tilde{P}_2^{\alpha}(u,v)\right) = R_2\left(P_2(u,v)\right). \tag{85}
$$

Now we calculate

$$
\begin{aligned}
R_3\left(\tilde{P}_2^{\alpha}(u,v)\right) &= \left(ua(v) - \alpha_1 a(v) - ua(\alpha_2), \frac{u^2}{2} - u\alpha_1 + F(v) - va(\alpha_2)\right)\\
&= R_3\left(P_2(u,v)\right) - \alpha_1 R_2\left(P_2(u,v)\right) - a(\alpha_2) R_1\left(P_2(u,v)\right)
\end{aligned} \tag{86}
$$

and

$$
\begin{aligned}
R_4\left(\tilde{P}_2^{\alpha}(u,v)\right) &= \left(\frac{u^2}{2} - u\alpha_1 - F(v) + va(v) - \alpha_2 a(v), uv - \alpha_1 v - u\alpha_2\right)\\
&= R_4\left(P_2(u,v)\right) - \alpha_1 R_1\left(P_2(u,v)\right) - \alpha_2 R_2\left(P_2(u,v)\right).
\end{aligned} \tag{87}
$$

This proves (83). Conversely, we have

$$
R_3\left(P_2(u,v)\right) \overset{(86),(85)}{=} R_3\left(\tilde{P}_2^{\alpha}(u,v)\right) + \alpha_1 R_2\left(\tilde{P}_2^{\alpha}(u,v)\right) + a(\alpha_2) R_1\left(\tilde{P}_2^{\alpha}(u,v)\right)
$$

and

$$
R_4\left(P_2(u,v)\right) \overset{(87),(85)}{=} R_4\left(\tilde{P}_2^{\alpha}(u,v)\right) + \alpha_1 R_1\left(\tilde{P}_2^{\alpha}(u,v)\right) + \alpha_2 R_2\left(\tilde{P}_2^{\alpha}(u,v)\right),
$$

and therefore we have (84). □

Proof of Lemma 15. By Lemma 16, it suffices to show that

$$
(P_2)_{\sharp}\nu \in \mathcal{M}^{pc}(\mathcal{K}_2) \Longleftrightarrow (\tilde{P}_2^{\alpha})_{\sharp}\nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^{\alpha}). \tag{88}
$$

We first show the implication "$\Longrightarrow$". We denote $\mu := (P_2)_{\sharp}\nu$ and $\mu^{\alpha} := (\tilde{P}_2^{\alpha})_{\sharp}\nu$, and assume $\mu \in \mathcal{M}^{pc}(\mathcal{K}_2)$. Recall the notation M_{ij} given by (69), and denote $\overline{\zeta} := \int_{\mathcal{K}_2} \zeta \, d\mu(\zeta)$. Then by the change of variable formula for push forward measures, Lemma 17 and bilinearity of the minor we have

$$\begin{aligned}\int_{\tilde{\mathcal{K}}_2^\alpha} M_{ij}(\zeta)\, d\mu^\alpha &= \int_{\mathbb{R}^2} M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \\ &= \int_{\mathbb{R}^2} \sum_{i',j'=1}^{4} c_{ii'}c_{jj'}\, M_{i'j'}\left(P_2(u,v)\right)\, d\nu \\ &= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} \int_{\mathcal{K}_2} M_{i'j'}(\zeta)\, d\mu \\ &\overset{\mu\in\mathcal{M}^{pc}(\mathcal{K}_2)}{=} \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}(\overline{\zeta}).\end{aligned} \tag{89}$$

On the other hand bilinearity of the minor implies that

$$\begin{aligned} M_{ij}\left(\int_{\tilde{\mathcal{K}}_2^\alpha} \zeta\, d\mu^\alpha\right) &= M_{ij}\left(\int_{\mathbb{R}^2} \tilde{P}_2^\alpha(u,v)\, d\nu\right) \\ &\overset{(69)}{=} \int_{\mathbb{R}^2} R_i\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \wedge \int_{\mathbb{R}^2} R_j\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \\ &= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} \int_{\mathbb{R}^2} R_{i'}\left(P_2(u,v)\right)\, d\nu \wedge \int_{\mathbb{R}^2} R_{j'}\left(P_2(u,v)\right)\, d\nu \\ &= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}\left(\int_{\mathbb{R}^2} P_2(u,v)\, d\nu\right) = \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}\left(\int_{\mathcal{K}_2} \zeta\, d\mu\right) \\ &= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}(\overline{\zeta}) \overset{(89)}{=} \int_{\tilde{\mathcal{K}}_2^\alpha} M_{ij}(\zeta)\, d\mu^\alpha \end{aligned}$$

as desired. The proof of the converse implication is analogous using (84). This completes the proof of (88), and hence Lemma 15. □

7. Existence of nontrivial measure in $\mathcal{M}^{pc}(\mathcal{K}_1)$

In this section, we first construct nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$ in the case $a'(\tilde{\alpha}_2) > 0$. Then it follows from Lemma 15 that we also have nontrivial elements in $\mathcal{M}^{pc}(\mathcal{K}_1)$. More precisely, we construct nontrivial measures supported at five points that belong to the space $\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$. To begin with, given $s_0, t_0 > 0$, recalling (71), we set

$$\zeta_0 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ \zeta_1 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1 + s_0, \tilde{\alpha}_2) = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix},$$

$$\zeta_2 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1 - s_0, \tilde{\alpha}_2) = \begin{pmatrix} -s_0 & 0 \\ 0 & -s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix},$$

$$\zeta_3 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2 + t_0) = \begin{pmatrix} 0 & t_0 \\ a(\tilde{\alpha}_2 + t_0) - a(\tilde{\alpha}_2) & 0 \\ 0 & F(\tilde{\alpha}_2 + t_0) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t_0 \end{pmatrix},$$

and

$$\zeta_4 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2 - t_0) = \begin{pmatrix} 0 & -t_0 \\ a(\tilde{\alpha}_2 - t_0) - a(\tilde{\alpha}_2) & 0 \\ 0 & F(\tilde{\alpha}_2 - t_0) - F(\tilde{\alpha}_2) + a(\tilde{\alpha}_2)t_0 \end{pmatrix}.$$

We first prove

Theorem 18. *Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, there exists $0 < \epsilon_0 < 1$ depending on the function a, $\tilde{\alpha}_2$, s_0 and t_0 such that, for all $\epsilon \leq \epsilon_0$, there exists a collection of weights $\{\gamma_j^\epsilon\}_{j=0}^4 \subset \mathbb{R}^+$ with $\sum_{j=1}^4 \gamma_j^\epsilon = \epsilon$ and $\gamma_0^\epsilon = 1 - \epsilon$ such that*

$$\mu^\epsilon := \sum_{j=0}^4 \gamma_j^\epsilon \delta_{\zeta_j} \in \mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}}).$$

The proof of Theorem 18 will rely on a couple of crucial lemmas. Let us first introduce some notations. We denote by D_1, D_2, D_3 the $(1,2), (2,3), (1,3)$ minors of a 3×2 matrix, respectively. We set the matrix

$$A := \begin{pmatrix} D_1(\zeta_1) & D_1(\zeta_2) & D_1(\zeta_3) & D_1(\zeta_4) \\ D_2(\zeta_1) & D_2(\zeta_2) & D_2(\zeta_3) & D_2(\zeta_4) \\ D_3(\zeta_1) & D_3(\zeta_2) & D_3(\zeta_3) & D_3(\zeta_4) \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{90}$$

For any $\epsilon > 0$ and $\gamma \in \mathbb{R}^4$, define

$$L^\epsilon(\gamma) := A\gamma - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}, \tag{91}$$

$$Q(\gamma) := \begin{pmatrix} D_1\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ D_2\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ D_3\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ 0 \end{pmatrix},$$

and

$$G^\epsilon(\gamma) := L^\epsilon(\gamma) - Q(\gamma). \tag{92}$$

Lemma 19. *Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, the matrix A defined in (90) is invertible. Moreover, for any $0 < \epsilon < 1$, the unique solution γ_0^ϵ of the system*

$$L^\epsilon(\gamma) = 0 \tag{93}$$

is nonnegative componentwise. Further, there exist constants $0 < \lambda < \Lambda < \infty$ depending on the function a, $\tilde{\alpha}_2$, s_0 and t_0 such that

$$\lambda\epsilon \leq [\gamma_0^\epsilon]_i \leq \Lambda\epsilon \ \text{ for } \ i = 1,2,3,4. \tag{94}$$

Proof. To simplify notation define $a_{\tilde{\alpha}_2}(t) := a(\tilde{\alpha}_2 + t) - a(\tilde{\alpha}_2)$ and $F_{\tilde{\alpha}_2}(t) := F(\tilde{\alpha}_2 + t) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t$. First, explicit calculations using the formulas for ζ_j, $j = 1,2,3,4$, give

$$A = \begin{pmatrix} s_0^2 & s_0^2 & -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ 0 & 0 & a_{\tilde{\alpha}_2}(t_0) F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0) F_{\tilde{\alpha}_2}(-t_0) \\ \frac{1}{2}s_0^3 & -\frac{1}{2}s_0^3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{95}$$

We claim that for any $(y_1, y_2, y_3) \neq 0 \in \mathbb{R}^3$, we have

$$\min_j \left\{ \sum_{i=1}^{3} y_i D_i(\zeta_j) \right\} < 0 \tag{96}$$

and

$$\max_j \left\{ \sum_{i=1}^{3} y_i D_i(\zeta_j) \right\} > 0. \tag{97}$$

We check (96) by an enumerative argument. Note that since $a'(\tilde{\alpha}_2) > 0$, assuming $t_0 > 0$ is small enough, we have that

$$a_{\tilde{\alpha}_2}(t_0) > 0 \text{ and } a_{\tilde{\alpha}_2}(-t_0) < 0. \tag{98}$$

We also know that F is convex in small neighborhood of $\tilde{\alpha}_2$ and so

$$F_{\tilde{\alpha}_2}(t_0) = F(\tilde{\alpha}_2 + t_0) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t_0 > 0 \text{ and } F_{\tilde{\alpha}_2}(-t_0) = F(\tilde{\alpha}_2 - t_0) - F(\tilde{\alpha}_2) + a(\tilde{\alpha}_2)t_0 > 0. \tag{99}$$

By carefully checking out the columns of A and using (98), (99) we see that

(1) If $y_1 > 0, y_2 \geq 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_4) < 0$.
(2) If $y_1 > 0, y_2 \leq 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_3) < 0$.
(3) If $y_1 < 0, y_3 \geq 0, y_2 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_2) < 0$.
(4) If $y_1 < 0, y_3 \leq 0, y_2 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_1) < 0$.
(5) If $y_1 = 0$, and
  (a) $y_2 > 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_4) = y_2 D_2(\zeta_4) < 0$;
  (b) $y_2 < 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_3) = y_2 D_2(\zeta_3) < 0$;
  (c) $y_2 = 0$, $y_3 > 0$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_2) = y_3 D_3(\zeta_2) < 0$;
  (d) $y_2 = 0$, $y_3 < 0$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_1) = y_3 D_3(\zeta_1) < 0$.

The above (1), (2) cover all cases when $y_1 > 0$, and (3), (4) cover all cases when $y_1 < 0$. For the case $y_1 = 0$, we have either $y_2 \neq 0$ or $y_2 = 0$. The former is covered by (5a), (5b), and the latter is covered by (5c), (5d). Therefore the above enumerative argument shows (96), and (97) is equivalent to (96). As we will briefly sketch, similar to the arguments in Lemma 11 using the Farkas-Minkowski Lemma, condition (96) guarantees that the system (93) has a solution that is nonnegative componentwise. Specifically, let a_1, a_2, a_3, a_4 denote the columns of the matrix A. Given any $y \in \mathbb{R}^4$ such that $y \cdot a_i \geq 0$ for any $i = 1, 2, 3, 4$, we must have that $y_4 > 0$. Hence the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}$ must be in the cone $\{\sum_{i=1}^{4} \lambda_i a_i : \lambda_i \in \mathbb{R}_+\}$. We deduce from the Farkas-Minkowski Lemma that (93) has a nonnegative solution.

To see the matrix A is invertible, consider the following system

$$A^T y = 0. \tag{100}$$

We claim that the system (100) has only the trivial solution. Indeed, let $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ be a solution of (100), i.e.,

$$\sum_{i=1}^{3} y_i D_i(\zeta_j) + y_4 = 0 \text{ for } j = 1, 2, 3, 4.$$

It follows from (96) and (97) that $y_4 = 0$, and therefore by (96) and (97) again, we have $y_i = 0$ for $i = 1, 2, 3$. It follows that A^T, and hence A, are invertible.

Finally, we show (94). Let γ_0^ϵ be the unique solution of (93). We already know that γ_0^ϵ is nonnegative componentwise. We will show that all components of γ_0^ϵ are strictly positive.

We argue by contradiction. Suppose $[\gamma_0^\epsilon]_1 = 0$ or $[\gamma_0^\epsilon]_2 = 0$. Then using the third row of (93), we have $\frac{s_0^3}{2}([\gamma_0^\epsilon]_1 - [\gamma_0^\epsilon]_2) \overset{(95)}{=} 0$ and therefore we have $[\gamma_0^\epsilon]_1 = [\gamma_0^\epsilon]_2 = 0$. Now using the first two rows of (93) (see (95)), we have

$$\begin{pmatrix} -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ a_{\tilde{\alpha}_2}(t_0)F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0)F_{\tilde{\alpha}_2}(-t_0) \end{pmatrix} \begin{pmatrix} [\gamma_0^\epsilon]_3 \\ [\gamma_0^\epsilon]_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is clear from (98), (99) that the matrix $\begin{pmatrix} -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ a_{\tilde{\alpha}_2}(t_0)F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0)F_{\tilde{\alpha}_2}(-t_0) \end{pmatrix}$ is invertible, and hence $[\gamma_0^\epsilon]_3 = [\gamma_0^\epsilon]_4 = 0$. But now we have $\gamma_0^\epsilon = 0$, which contradicts the fourth row of (93). This contradiction implies $[\gamma_0^\epsilon]_1 > 0$ and $[\gamma_0^\epsilon]_2 > 0$. A similar argument yields $[\gamma_0^\epsilon]_3 > 0$ and $[\gamma_0^\epsilon]_4 > 0$. Since $[\gamma_0^\epsilon]_i = \epsilon [A^{-1}]_{i4}$, $i = 1,2,3,4$, it follows that $[A^{-1}]_{i4} > 0$ for all i. Now we define

$$\lambda := \min_i \{[A^{-1}]_{i4}\} \quad \text{and} \quad \Lambda := \max_i \{[A^{-1}]_{i4}\}.$$

It is clear that $0 < \lambda \leq \Lambda < \infty$ and (94) is satisfied. Note that A^{-1} is a fixed matrix independent of ϵ, and so are λ and Λ independent of ϵ. ☐

Lemma 20. *Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, there exists $0 < \epsilon_0 < 1$ sufficiently small such that for all $0 < \epsilon \leq \epsilon_0$, the system*

$$G^\epsilon(\gamma) = 0 \tag{101}$$

has a nonnegative solution.

Proof. Given a sufficiently small $0 < \epsilon < 1$ whose size will be specified later, by Lemma 19, the linear system $L^\epsilon(\gamma) = 0$ has a unique nonnegative solution γ_0^ϵ that satisfies the estimate (94). We will find a solution to (101) by iteration. For all $k \in \mathbb{N}^+$, define

$$\Delta_k^\epsilon := A^{-1}\left(-G^\epsilon(\gamma_{k-1}^\epsilon)\right) \text{ and } \gamma_k^\epsilon := \gamma_{k-1}^\epsilon + \Delta_k^\epsilon. \tag{102}$$

Then we have

$$\begin{aligned} G^\epsilon(\gamma_k^\epsilon) &\overset{(92)}{=} L^\epsilon(\gamma_k^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(91),(102)}{=} A\left(\gamma_{k-1}^\epsilon + \Delta_k^\epsilon\right) - (0,0,0,\epsilon)^T - Q(\gamma_k^\epsilon) \\ &\overset{(91)}{=} L^\epsilon(\gamma_{k-1}^\epsilon) + A(\Delta_k^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(92)}{=} G^\epsilon\left(\gamma_{k-1}^\epsilon\right) + A\left(\Delta_k^\epsilon\right) + Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(102)}{=} G^\epsilon\left(\gamma_{k-1}^\epsilon\right) - G^\epsilon\left(\gamma_{k-1}^\epsilon\right) + Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon) \\ &= Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon). \end{aligned} \tag{103}$$

Now let us estimate the sizes of Δ_k^ϵ and γ_k^ϵ. First note that $D_i\left(\sum_{j=1}^4 \gamma_j \zeta_j\right)$, $i = 1,2,3$, is a fixed quadratic function of γ whose coefficients depend only on the function a, $\tilde{\alpha}_2$, s_0 and t_0. Therefore, for all $r > 0$ and $\gamma, \tilde{\gamma} \in B_r(0) \subset \mathbb{R}^4$, we have

$$\|Q(\gamma)\| \leq C_1 \|\gamma\|^2 \tag{104}$$

and

$$\|Q(\gamma) - Q(\tilde{\gamma})\| \leq \sup_{z \in B_r(0)} \|DQ(z)\| \cdot \|\gamma - \tilde{\gamma}\| \leq C_1 r \|\gamma - \tilde{\gamma}\|, \tag{105}$$

where the above constant C_1 depends only on the coefficients of Q and therefore does not depend on ϵ or $\gamma, \tilde{\gamma}, r$. Let $\theta > 0$ be sufficiently small such that

$$\sum_{p=1}^{\infty} 2^{p-1}\theta^p \leq \frac{\lambda}{4\Lambda} < 1. \tag{106}$$

Clearly such θ exists. We denote

$$C_2 := \left\| A^{-1} \right\| C_1. \tag{107}$$

Let $\epsilon_0 := \frac{\theta}{2C_2\Lambda} > 0$. Now for all $0 < \epsilon \leq \epsilon_0$, it follows from (94) that

$$C_2 \|\gamma_0^\epsilon\| \leq C_2 \cdot 2\Lambda\epsilon_0 = \theta. \tag{108}$$

We claim that

$$\|\Delta_k^\epsilon\| \leq 2^{k-1}\theta^k \|\gamma_0^\epsilon\| \tag{109}$$

and

$$\|\gamma_k^\epsilon\| \leq \left(1 + \sum_{p=1}^{k} 2^{p-1}\theta^p\right) \|\gamma_0^\epsilon\| < 2\|\gamma_0^\epsilon\|. \tag{110}$$

We show this by induction. Recall that $L^\epsilon(\gamma_0^\epsilon) = 0$. We deduce from (104) that

$$\|-G^\epsilon(\gamma_0^\epsilon)\| \overset{(92)}{=} \|Q(\gamma_0^\epsilon)\| \overset{(104)}{\leq} C_1 \|\gamma_0^\epsilon\|^2. \tag{111}$$

It follows from this, (102) and (108) that

$$\|\Delta_1^\epsilon\| \overset{(102),(111)}{\leq} \left\| A^{-1} \right\| \cdot C_1 \|\gamma_0^\epsilon\|^2 = C_2 \|\gamma_0^\epsilon\|^2 \overset{(108)}{\leq} \theta \|\gamma_0^\epsilon\| \tag{112}$$

and therefore

$$\|\gamma_1^\epsilon\| \overset{(102)}{\leq} (1+\theta) \|\gamma_0^\epsilon\|. \tag{113}$$

So by (112), (113) we have that (109), (110) hold for $k = 1$. Now suppose (109), (110) hold for $k \geq 1$. Using (103), (105) and the induction assumption, we have

$$\begin{aligned} \|G^\epsilon(\gamma_k^\epsilon)\| &\overset{(103)}{=} \|Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon)\| \\ &\overset{(105),(110),(102)}{\leq} C_1 \cdot 2\|\gamma_0^\epsilon\| \cdot \|\Delta_k^\epsilon\| \overset{(109)}{\leq} C_1 2^k \theta^k \|\gamma_0^\epsilon\|^2. \end{aligned} \tag{114}$$

It follows from (102) and (108) that

$$\left\|\Delta_{k+1}^\epsilon\right\| \overset{(102)}{\leq} \left\| A^{-1} \right\| \cdot \|G^\epsilon(\gamma_k^\epsilon)\| \overset{(114),(107)}{\leq} C_2 2^k \theta^k \|\gamma_0^\epsilon\|^2 \overset{(108)}{\leq} 2^k \theta^{k+1} \|\gamma_0^\epsilon\| \tag{115}$$

and

$$\left\|\gamma_{k+1}^\epsilon\right\| \overset{(102)}{\leq} \|\gamma_0^\epsilon\| + \sum_{p=1}^{k+1} \left\|\Delta_p^\epsilon\right\| \overset{(115)}{\leq} \left(1 + \sum_{p=1}^{k+1} 2^{p-1}\theta^p\right) \|\gamma_0^\epsilon\| \overset{(106)}{\leq} 2\|\gamma_0^\epsilon\|.$$

Thus we have established (109), (110) for general k.

Since $\{\gamma_k^\epsilon\}_k$ forms a bounded sequence, it has a convergent subsequence such that (without relabeling)

$$\lim_{k\to\infty} \gamma_k^\epsilon = \bar{\gamma}^\epsilon$$

for some $\bar{\gamma}^\epsilon$. We claim that $\bar{\gamma}^\epsilon$ is a nonnegative solution to (101). From the estimates (114) and (106), we have

$$\|G^\epsilon(\gamma_k^\epsilon)\| \overset{(114)}{\leq} C_1 2^k \theta^k \|\gamma_0^\epsilon\|^2 \to 0 \quad \text{as} \quad k \to \infty.$$

Since G^ϵ is continuous, we have

$$\|G^\epsilon(\bar\gamma^\epsilon)\| = \lim_{k\to\infty}\|G^\epsilon(\gamma_k^\epsilon)\| = 0.$$

It only remains to show that $\bar\gamma^\epsilon$ is nonnegative componentwise. We deduce from (109), (94) and (106) that

$$\|\gamma_k^\epsilon - \gamma_0^\epsilon\| \overset{(102)}{\leq} \sum_{p=1}^{k}\left\|\Delta_p^\epsilon\right\| \overset{(109)}{\leq} \sum_{p=1}^{k} 2^{p-1}\theta^p\,\|\gamma_0^\epsilon\| \overset{(106),(94)}{\leq} \frac{\lambda}{4\Lambda}\cdot 2\Lambda\epsilon = \frac{\lambda}{2}\epsilon. \tag{116}$$

We know from (94) that each component of γ_0^ϵ is bounded below by $\lambda\epsilon$. This together with (116) shows that all components of γ_k^ϵ are bounded below by $\frac{\lambda}{2}\epsilon$ for all k. Therefore, the same holds for $\bar\gamma^\epsilon$. In particular, $\bar\gamma^\epsilon$ is nonnegative. □

Proof of Theorem 18. Given $0<\epsilon\leq\epsilon_0<1$, let $\bar\gamma^\epsilon=(\bar\gamma_1^\epsilon,\bar\gamma_2^\epsilon,\bar\gamma_3^\epsilon,\bar\gamma_4^\epsilon)$ be the nonnegative solution of (101) found in Lemma 20. Then we have $\sum_{j=1}^4\bar\gamma_j^\epsilon=\epsilon$. Define $\bar\gamma_0^\epsilon:=1-\epsilon$. Then we have $\bar\gamma_j^\epsilon\geq 0$ for all $j=0,1,2,3,4$ and $\sum_{j=0}^4\bar\gamma_j^\epsilon=1$. Now we define

$$\mu^\epsilon := \sum_{j=0}^{4}\bar\gamma_j^\epsilon\delta_{\zeta_j}.$$

It is clear that μ^ϵ is a probability measure. Since $0<\epsilon<1$, μ^ϵ is nontrivial. Since ζ_0 is the trivial matrix and $\bar\gamma^\epsilon$ solves the system (101), we have

$$\sum_{j=0}^{4}\bar\gamma_j^\epsilon D_i(\zeta_j) = \sum_{j=1}^{4}\bar\gamma_j^\epsilon D_i(\zeta_j) \overset{(101),(92)}{=} D_i\left(\sum_{j=1}^{4}\bar\gamma_j^\epsilon\zeta_j\right) = D_i\left(\sum_{j=0}^{4}\bar\gamma_j^\epsilon\zeta_j\right)$$

for all $i=1,2,3$. This shows that $\mu^\epsilon\in\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde\alpha})$. □

Proof of Theorem 4 completed. We first consider the case $a'(\tilde\alpha_2)>0$. Given $0<\epsilon\leq\epsilon_0<1$, let $\mu^\epsilon\in\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde\alpha})$ be the measure constructed in Theorem 18. Let $\nu^\epsilon := \left((P_1^{\tilde\alpha})^{-1}\right)_\sharp\mu^\epsilon$. Note that since $P_1^{\tilde\alpha}$ is a bijection, we have $\mu^\epsilon=(P_1^{\tilde\alpha})_\sharp\nu^\epsilon$. Define $\tilde\mu^\epsilon:=(P_1)_\sharp\nu^\epsilon$. Since $(P_1^{\tilde\alpha})_\sharp\nu^\epsilon = \mu^\epsilon\in\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde\alpha})$, it follows from Lemma 15 that $\tilde\mu^\epsilon=(P_1)_\sharp\nu^\epsilon\in\mathcal{M}^{pc}(\mathcal{K}_1)$. Since P_1 and $P_1^{\tilde\alpha}$ are both bijections, it is clear that $\tilde\mu^\epsilon$ is also supported at five points, and hence is nontrivial. Further, by choosing s_0,t_0 sufficiently small in Theorem 18, one can make the support of μ^ϵ sufficiently small. It follows from Lemma 14 that the support of $\tilde\mu^\epsilon$ can be made sufficiently small. This establishes the case where $a'(\tilde\alpha_2)>0$.

Now suppose $a'(\tilde\alpha_2)<0$, then for some $\delta>0$ sufficiently small we have that

$$(v_2-v_1)(a(v_2)-a(v_1))<0 \text{ for any } v_1,v_2\in(a(\tilde\alpha_2)-\delta,a(\tilde\alpha_2)+\delta). \tag{117}$$

Let $\mathcal{K}_0:=\left\{\begin{pmatrix} u & v\\ a(v) & u\end{pmatrix}: u,v\in\mathbb{R}\right\}$. Note that if $\det\begin{pmatrix} u_2-u_1 & v_2-v_1\\ a(v_2)-a(v_1) & u_2-u_1\end{pmatrix}=0$ for some (u_1,v_1) and (u_2,v_2) in $B_\delta(P_1(\tilde\alpha))$, then

$$(u_2-u_1)^2-(v_2-v_1)(a(v_2)-a(v_1))=0,$$

which by (117) implies $u_1=u_2$ and $v_1=v_2$. Thus, for sufficiently small neighborhood $\tilde U$ of $P_1(\tilde\alpha)$, $\mathcal{K}_0\cap\tilde U$ does not contain Rank-1 connections and $\det(X-Y)$ does not change sign on $(\mathcal{K}_0\cap\tilde U)\times(\mathcal{K}_0\cap\tilde U)$. By [Sv 93] Lemma 3 we have that $\mathcal{M}^{pc}(\mathcal{K}_0\cap\tilde U)$ consists of Dirac measures only. As $\mathcal{M}^{pc}(\mathcal{K}_1\cap U)$ can be embedded in $\mathcal{M}^{pc}(\mathcal{K}_0\cap\tilde U)$, this completes the proof of the case $a'(\tilde\alpha_2)<0$, and hence the proof of Theorem 4. □

8. A more direct proof of DiPerna's theorem

We will present a direct proof of triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$ (Theorem 6) motivated by the perspective of searching for linear combinations of minors that are nonnegative. As a simple consequence, we give a proof of DiPerna's Theorem 5.

8.1. **Proof of Theorem 6.** The proof of Theorem 6 requires a couple of auxiliary lemmas.

Lemma 21. *For all $\alpha \in \mathbb{R}^2$ such that $a'(\alpha_2) > 0$ and $a''(\alpha_2) \neq 0$, there exists $\delta_1 > 0$ depending on α and the function a such that for all $(u,v) \in \mathbb{R}^2$ with $(u,v) \in B_{\delta_1}(\alpha)$, we have*

$$\begin{aligned} &- M_{34}(P_2^\alpha(u,v)) - \frac{2a'(\alpha_2)}{a''(\alpha_2)} M_{23}(P_2^\alpha(u,v)) - \frac{2(a'(\alpha_2))^2}{a''(\alpha_2)} M_{14}(P_2^\alpha(u,v)) \\ &\geq \frac{1}{8}\left|u-\alpha_1\right|^4 + \frac{a'(\alpha_2)^2}{24}\left|v-\alpha_2\right|^4. \end{aligned} \tag{118}$$

Proof. Let us denote $s := u - \alpha_1$, $t := v - \alpha_2$. Then using the definition of P_2^α in (72) we have

$$P_2^\alpha(u,v) = \begin{pmatrix} s & t \\ a(t+\alpha_2) - a(\alpha_2) & s \\ s\left(a(t+\alpha_2) - a(\alpha_2)\right) & \frac{1}{2}s^2 + F(t+\alpha_2) - F(\alpha_2) - a(\alpha_2)t \\ \frac{1}{2}s^2 + F(\alpha_2) - F(t+\alpha_2) + a(t+\alpha_2)t & st \end{pmatrix}.$$

Step 1. We write out the Taylor expansion of the entries of $P_2^\alpha(u,v)$ and calculate the Taylor expansion of the minors up to order four. This idea of looking at terms up to order four comes from the work of DiPerna [DP 85]. First we have

$$\begin{aligned} a(\alpha_2+t) - a(\alpha_2) &\approx a'(\alpha_2)t + \frac{a''(\alpha_2)}{2}t^2 + \frac{a'''(\alpha_2)}{6}t^3, \\ s\left(a(\alpha_2+t) - a(\alpha_2)\right) &\approx a'(\alpha_2)st + \frac{a''(\alpha_2)}{2}st^2, \\ F(\alpha_2+t) - F(\alpha_2) - a(\alpha_2)t &\approx \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{6}t^3, \end{aligned} \tag{119}$$

and

$$\begin{aligned} F(\alpha_2) - F(\alpha_2+t) + a(\alpha_2+t)t &= -\left(F(\alpha_2+t) - F(\alpha_2) - a(\alpha_2)t\right) + \left(a(\alpha_2+t) - a(\alpha_2)\right)t \\ &\overset{(119)}{\approx} -\frac{a'(\alpha_2)}{2}t^2 - \frac{a''(\alpha_2)}{6}t^3 + a'(\alpha_2)t^2 + \frac{a''(\alpha_2)}{2}t^3 \\ &= \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{3}t^3. \end{aligned}$$

In the above calculations, we have omitted terms that are of order four or higher. Putting the above calculations together, we obtain

$$P_2^\alpha(u,v) \approx \begin{pmatrix} s & t \\ a'(\alpha_2)t + \frac{a''(\alpha_2)}{2}t^2 + \frac{a'''(\alpha_2)}{6}t^3 & s \\ a'(\alpha_2)st + \frac{a''(\alpha_2)}{2}st^2 & \frac{s^2}{2} + \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{6}t^3 \\ \frac{s^2}{2} + \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{3}t^3 & st \end{pmatrix}. \tag{120}$$

Step 2. We denote by $M_{jk}^p\left(P_2^\alpha(u,v)\right)$ the p-th order terms in the minor $M_{jk}\left(P_2^\alpha(u,v)\right)$, and we calculate the $(1,4), (2,3)$ and $(3,4)$ minors of (120) up to order four.

$$\begin{aligned} M_{14}^2\left(P_2^\alpha(u,v)\right) &= 0, \\ M_{14}^3\left(P_2^\alpha(u,v)\right) &= s^2t - \frac{1}{2}s^2t - \frac{a'(\alpha_2)}{2}t^3 = \frac{1}{2}s^2t - \frac{a'(\alpha_2)}{2}t^3, \end{aligned}$$

$$M^4_{14}\left(P^{\alpha}_2(u,v)\right)=-\frac{a''(\alpha_2)}{3}t^4,$$

$$M^2_{23}\left(P^{\alpha}_2(u,v)\right)=0,$$
$$M^3_{23}\left(P^{\alpha}_2(u,v)\right)=\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3-a'(\alpha_2)s^2t=-\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3,$$
$$M^4_{23}\left(P^{\alpha}_2(u,v)\right)=\frac{a'(\alpha_2)a''(\alpha_2)}{6}t^4+\frac{a''(\alpha_2)}{2}t^2\left(\frac{s^2}{2}+\frac{a'(\alpha_2)}{2}t^2\right)-\frac{a''(\alpha_2)}{2}s^2t^2$$
$$=\frac{5}{12}a'(\alpha_2)a''(\alpha_2)t^4-\frac{a''(\alpha_2)}{4}s^2t^2,$$
$$M^2_{34}\left(P^{\alpha}_2(u,v)\right)=M^3_{34}\left(P^{\alpha}_2(u,v)\right)=0,$$

$$M^4_{34}\left(P^{\alpha}_2(u,v)\right)=a'(\alpha_2)s^2t^2-\left(\frac{s^2}{2}+\frac{a'(\alpha_2)}{2}t^2\right)^2=-\frac{1}{4}s^4+\frac{a'(\alpha_2)}{2}s^2t^2-\frac{a'(\alpha_2)^2}{4}t^4.$$

Step 3. We show (118). To see this, let us denote by

$$R^p:=-M^p_{34}(P^{\alpha}_2(u,v))-\frac{2a'(\alpha_2)}{a''(\alpha_2)}M^p_{23}(P^{\alpha}_2(u,v))-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}M^p_{14}(P^{\alpha}_2(u,v))$$

for $p=2,3,4$. From the calculations in Step 2, we have

$$R^2=0,$$
$$R^3=-\frac{2a'(\alpha_2)}{a''(\alpha_2)}\left(-\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3\right)-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}\left(\frac{s^2t}{2}-\frac{a'(\alpha_2)}{2}t^3\right)=0,$$
$$R^4=\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}\frac{a''(\alpha_2)}{3}t^4-\frac{2a'(\alpha_2)}{a''(\alpha_2)}\left(\frac{5}{12}a'(\alpha_2)a''(\alpha_2)t^4-\frac{a''(\alpha_2)}{4}s^2t^2\right)$$
$$-\left(-\frac{1}{4}s^4+\frac{a'(\alpha_2)}{2}s^2t^2-\frac{a'(\alpha_2)^2}{4}t^4\right)=\frac{1}{4}s^4+\frac{a'(\alpha_2)^2}{12}t^4.$$

Therefore, we have

$$-M_{34}(P^{\alpha}_2(u,v))-\frac{2a'(\alpha_2)}{a''(\alpha_2)}M_{23}(P^{\alpha}_2(u,v))-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}M_{14}(P^{\alpha}_2(u,v))=\frac{1}{4}s^4+\frac{a'(\alpha_2)^2}{12}t^4+\text{l.o.t.},$$

where

$$\text{l.o.t.}\le C\sum_{i+j=5}s^it^j$$

for some constant C. Hence, there exists $\delta_1>0$ sufficiently small such that (118) holds for all $(s,t)\in B_{\delta_1}(0)$. □

Lemma 22. *Let $\nu\in\mathcal{P}(\mathbb{R}^2)$ satisfy $(P_2)_\sharp\nu\in\mathcal{M}^{pc}(\mathcal{K}_2)$. Recall the notations $\overline{u}=\int ud\nu$ and $\overline{v}=\int vd\nu$. Assume $a'(\overline{v})>0$ and $a''(\overline{v})\neq 0$. In addition assume that*

$$\text{Spt}\nu\subset B_{\delta_2}((\overline{u},\overline{v}))\tag{121}$$

for some $\delta_2>0$ sufficiently small. Recall the definition of P^{α}_2 in (72), and define

$$\overline{P}=\int P^{(\overline{u},\overline{v})}_2(u,v)d\nu.\tag{122}$$

We have

$$-M_{34}(\overline{P})-\frac{2a'(\overline{v})}{a''(\overline{v})}M_{23}(\overline{P})-\frac{2a'(\overline{v})^2}{a''(\overline{v})}M_{14}(\overline{P})\le C\delta_2\int|v-\overline{v}|^4\,d\nu.\tag{123}$$

Proof. Note that by Lemma 15 we have $(P_2^{(\overline{u},\overline{v})})_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2^{(\overline{u},\overline{v})})$. From the expression of $P_2^{(\overline{u},\overline{v})}$ it is clear that

$$\left[\overline{P}\right]_{11} = 0 \text{ and } \left[\overline{P}\right]_{12} = 0. \tag{124}$$

So we have

$$\begin{aligned}\int \left[(u-\overline{u})^2 + (a(v) - a(\overline{v}))(v-\overline{v})\right] d\nu &= \int M_{12}\left(P_2^{(\overline{u},\overline{v})}(u,v)\right) d\nu \\ &\overset{(75)}{=} M_{12}\left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu\right) \overset{(122)}{=} M_{12}(\overline{P}) \overset{(124)}{=} 0.\end{aligned} \tag{125}$$

Now by Taylor's theorem we have that $a(v) = a(\overline{v}) + a'(\overline{v})(v-\overline{v}) + \frac{a''(d_v)}{2}(v-\overline{v})^2$ for some $d_v \in [\overline{v}, v]$. So we deduce from (125) that

$$\left|\int \left[(u-\overline{u})^2 - a'(\overline{v})(v-\overline{v})^2\right] d\nu\right| \leq C\int |v-\overline{v}|^3\, d\nu. \tag{126}$$

Also by Taylor expansion we have

$$\left|F(\overline{v}) - F(v) - a(v)(\overline{v}-v) - \frac{a'(v)}{2}(\overline{v}-v)^2\right| \leq C\,|v-\overline{v}|^3. \tag{127}$$

As

$$\left|\int \left[\frac{a'(v)}{2}(v-\overline{v})^2 - \frac{a'(\overline{v})}{2}(v-\overline{v})^2\right] d\nu\right| \leq C\int |v-\overline{v}|^3\, d\nu, \tag{128}$$

putting (127) and (128) together we simplify $\left[\overline{P}\right]_{41}$ to obtain

$$\begin{aligned}&\left|\int \left[\frac{(u-\overline{u})^2}{2} + F(\overline{v}) - F(v) - a(v)(\overline{v}-v)\right] d\nu - \frac{1}{2}\int \left[(u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2\right] d\nu\right| \\ &\quad \leq C\int |v-\overline{v}|^3\, d\nu.\end{aligned} \tag{129}$$

By a very similar argument as above we obtain

$$\left|\int (a(v) - a(\overline{v}))\, d\nu - \int \frac{a''(\overline{v})}{2}(v-\overline{v})^2\, d\nu\right| \leq C\int |v-\overline{v}|^3\, d\nu, \tag{130}$$

$$\left|\int (u-\overline{u})(a(v) - a(\overline{v}))\, d\nu - \int a'(\overline{v})(u-\overline{u})(v-\overline{v})\, d\nu\right| \leq C\int \left|(u-\overline{u})(\overline{v}-v)^2\right| d\nu, \tag{131}$$

and

$$\begin{aligned}&\left|\int \left[\frac{(u-\overline{u})^2}{2} + F(v) - F(\overline{v}) - a(\overline{v})(v-\overline{v})\right] d\nu - \frac{1}{2}\int \left[(u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2\right] d\nu\right| \\ &\quad \leq C\int |v-\overline{v}|^3\, d\nu.\end{aligned} \tag{132}$$

Using (129)-(132) to clean up $\overline{P}$ we obtain

$$\overline{P} = \begin{pmatrix} 0 & 0 \\ \int \frac{a''(\overline{v})}{2}(v-\overline{v})^2\, d\nu + E_{21} & 0 \\ \int a'(\overline{v})(u-\overline{u})(v-\overline{v})\, d\nu + E_{31} & \int \frac{1}{2}\left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2\right) d\nu + E_{32} \\ \int \frac{1}{2}\left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2\right) d\nu + E_{41} & \int (u-\overline{u})(v-\overline{v})\, d\nu \end{pmatrix},$$

where

$$|E_{ij}| \leq C\left(\int \left|(u-\overline{u})(v-\overline{v})^2\right| d\nu + \int |v-\overline{v}|^3\, d\nu\right). \tag{133}$$

Let

$$\widetilde{P} := \begin{pmatrix} 0 & 0 \\ \int \frac{a''(\overline{v})}{2} (v-\overline{v})^2 \, d\nu & 0 \\ \int a'(\overline{v}) \, (u-\overline{u}) \, (v-\overline{v}) \, d\nu & \int \frac{1}{2} \left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2 \right) d\nu \\ \int \frac{1}{2} \left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2 \right) d\nu & \int (u-\overline{u}) \, (v-\overline{v}) \, d\nu \end{pmatrix}. \tag{134}$$

Using (121) we have

$$\sup \{ |(u,v) - (\overline{u},\overline{v})| : (u,v) \in \mathrm{Spt}\nu \} \leq \delta_2. \tag{135}$$

Now

$$\begin{aligned} \left| \int a'(\overline{v}) \, (u-\overline{u}) \, (v-\overline{v}) \, d\nu \right| &\leq C \left(\int (u-\overline{u})^2 \, d\nu \right)^{\frac{1}{2}} \left(\int |v-\overline{v}|^2 \, d\nu \right)^{\frac{1}{2}} \\ &\overset{(126)}{\leq} C \left(\int |v-\overline{v}|^2 \, d\nu + |v-\overline{v}|^3 \, d\nu \right)^{\frac{1}{2}} \left(\int |v-\overline{v}|^2 \, d\nu \right)^{\frac{1}{2}} \\ &\overset{(135)}{\leq} C \int |v-\overline{v}|^2 \, d\nu. \end{aligned}$$

In a similar way using (126) we have that

$$\left| \left[\widetilde{P} \right]_{ij} \right| \leq C \int |v-\overline{v}|^2 \, d\nu \text{ for any } 1 \leq i < j \leq 4. \tag{136}$$

It follows from (133) and (136) that for all $1 \leq i < j \leq 4$ we have

$$\left| M_{ij}(\overline{P}) - M_{ij}(\widetilde{P}) \right| \leq C \left(\int \left[|v-\overline{v}|^3 + |u-\overline{u}| \, |v-\overline{v}|^2 \right] d\nu \right) \left(\int |v-\overline{v}|^2 \, d\nu \right).$$

Using (135) and Hölder's inequality we further have that

$$\left| M_{ij}(\overline{P}) - M_{ij}(\widetilde{P}) \right| \leq C\delta_2 \int |v-\overline{v}|^4 \, d\nu. \tag{137}$$

It is clear from (126) that

$$\left| \int \frac{1}{2} \left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2 \right) d\nu - \int (u-\overline{u})^2 \, d\nu \right| \leq C \int |v-\overline{v}|^3 \, d\nu. \tag{138}$$

Therefore we have

$$\begin{aligned} & - M_{34}(\overline{P}) - \frac{2a'(\overline{v})}{a''(\overline{v})} M_{23}(\overline{P}) - \frac{2a'(\overline{v})^2}{a''(\overline{v})} M_{14}(\overline{P}) \\ &\overset{(134),(138)}{=} - \left(a'(\overline{v}) \left(\int (u-\overline{u}) \, (v-\overline{v}) \, d\nu \right)^2 - \frac{1}{4} \left(\int \left((u-\overline{u})^2 + a'(\overline{v})(v-\overline{v})^2 \right) d\nu \right)^2 \right) \\ &\quad - 2 \frac{a'(\overline{v})}{a''(\overline{v})} \left(\frac{a''(\overline{v})}{2} \int (v-\overline{v})^2 \, d\nu \right) \left(\int (u-\overline{u})^2 \, d\nu \right) + \text{Error terms} \\ &= -a'(\overline{v}) \left(\int (u-\overline{u}) \, (v-\overline{v}) \, d\nu \right)^2 + \frac{1}{4} \left(\int (u-\overline{u})^2 \; d\nu \right)^2 + \frac{(a'(\overline{v}))^2}{4} \left(\int (v-\overline{v})^2 \, d\nu \right)^2 \\ &\quad - \frac{a'(\overline{v})}{2} \left(\int (v-\overline{v})^2 \, d\nu \right) \left(\int (u-\overline{u})^2 \, d\nu \right) + \text{Error terms}, \end{aligned} \tag{139}$$

where we know by (137) that $|\text{Error terms}| \leq C\delta_2 \int |v-\overline{v}|^4 \, d\nu$. Since $a'(\overline{v}) > 0$, we have $-a'(\overline{v}) \left(\int (u-\overline{u})(v-\overline{v}) \, d\nu\right)^2 \leq 0$. Also note that from (126) we have

$$\begin{aligned}\frac{1}{4}\left(\int (u-\overline{u})^2 \, d\nu\right)^2 &+ \frac{(a'(\overline{v}))^2}{4}\left(\int (v-\overline{v})^2 \, d\nu\right)^2 \\ &- \frac{a'(\overline{v})}{2}\left(\int (v-\overline{v})^2 \, d\nu\right)\left(\int (u-\overline{u})^2 \, d\nu\right) \\ = \frac{1}{4}\left(\int (u-\overline{u})^2 \, d\nu - a'(\overline{v})\int (v-\overline{v})^2 \, d\nu\right)^2 &\leq C\left(\int |v-\overline{v}|^3 d\nu\right)^2.\end{aligned}$$

The above together with (135) and (139) gives (123). □

Proof of Theorem 6. Define $\tilde{\delta}_0 := \min\{\frac{\delta_1}{2}, \delta_2\}$. For $0 < \delta \leq \tilde{\delta}_0$, let $\mu \in \mathcal{M}^{pc}(\mathcal{K}_2 \cap B_\delta(P_2(\tilde{\alpha})))$. Define $\nu := \left((P_2)^{-1}\right)_\sharp \mu$ and let $\alpha = (\overline{u}, \overline{v})$. Using arguments similar to those in Lemma 14, we have $\mathrm{Spt}\nu \subset B_\delta(\tilde{\alpha})$. Also it is clear that $(\overline{u}, \overline{v}) \in B_\delta(\tilde{\alpha})$. Without loss of generality, we may assume $a'(\overline{v}) > 0$ and $a''(\overline{v}) \neq 0$, as otherwise we can reduce the size of $\tilde{\delta}_0$ until these are satisfied. Since $\delta \leq \tilde{\delta}_0 \leq \delta_2$, Lemma 22 applies. For all $(u,v) \in \mathrm{Spt}\nu$, we have $|(u,v) - (\overline{u},\overline{v})| \leq |(u,v) - \tilde{\alpha}| + |\tilde{\alpha} - (\overline{u},\overline{v})| \leq 2\delta \leq 2\tilde{\delta}_0 \leq \delta_1$. Hence, for all $(u,v) \in \mathrm{Spt}\nu$, the estimate (118) holds.

Let $\lambda_{34} = -1$, $\lambda_{23} = \frac{-2a'(\overline{v})}{a''(\overline{v})}$, $\lambda_{14} = \frac{-2(a'(\overline{v}))^2}{a''(\overline{v})}$. For any $1 \leq i < j \leq 4$ with $(i,j) \notin \{(3,4),(2,3),(1,4)\}$ define $\lambda_{ij} = 0$. Note that by Lemma 14 we have that

$$\int \sum_{i<j=1}^{4} \lambda_{ij} M_{ij}\left(P_2^{(\overline{u},\overline{v})}(u,v)\right) d\nu = \sum_{i<j=1}^{4} \lambda_{ij} M_{ij}\left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu\right).$$

By Lemma 21 we have that

$$\int \sum_{i<j=1}^{4} \lambda_{ij} M_{ij}\left(P_2^{(\overline{u},\overline{v})}(u,v)\right) d\nu \geq \int \left(\frac{|u-\overline{u}|^4}{8} + \frac{(a'(\overline{v}))^2}{24}|v-\overline{v}|^4\right) d\nu. \tag{140}$$

On the other hand by Lemma 22 we have that

$$\sum_{i,j=1}^{4} \lambda_{ij} M_{ij}\left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu\right) \leq C\delta \int |v-\overline{v}|^4 \, d\nu. \tag{141}$$

Putting (140) and (141) together we have that

$$\int \left(\frac{|u-\overline{u}|^4}{8} + \frac{a'(\overline{v})^2}{24}|v-\overline{v}|^4\right) d\nu \leq C\delta \int |v-\overline{v}|^4 \, d\nu. \tag{142}$$

Hence there exists $\delta_0 \leq \tilde{\delta}_0$ such that for all $0 < \delta \leq \delta_0$ we can absorb the right hand side of (142) into the left hand side and hence have

$$\int \left(\frac{|u-\overline{u}|^4}{8} + \frac{a'(\overline{v})^2}{48}|v-\overline{v}|^4\right) d\nu \leq 0.$$

It follows that $\nu = \delta_{(\overline{u},\overline{v})}$, and hence $\mu = (P_2)_\sharp \nu$ is also a Dirac measure. □

8.2. **Proof of Theorem 5.** Finally we give the proof of Theorem 5 as a consequence of Theorem 6.

Proof of Theorem 5. Let

$$(g_{1,1}(u,v), g_{1,2}(u,v)) := (v, -u) \text{ and } (g_{2,1}(u,v), g_{2,2}(u,v)) := (u, -a(v)). \tag{143}$$

Further, denote

$$(g_{i+2,1}(u,v), g_{i+2,2}(u,v)) := (\eta_i(u,v), q_i(u,v)) \text{ for } i = 1,2. \tag{144}$$

By hypothesis we have that

$$\operatorname{div}(g_{i,1}(u^\epsilon, v^\epsilon), g_{i,2}(u^\epsilon, v^\epsilon)) \text{ is precompact in } W^{-1,2}_{loc} \text{ for } i = 1,2,3,4.$$

Let $\{(u^{\epsilon_n}, v^{\epsilon_n})\}$ be a subsequence that generates the Young measure $\nu_{t,x}$ associated with the weak* convergence of $\{(u^\epsilon, v^\epsilon)\}$. Now for any $1 \le i < j \le 4$, by the fundamental theorem of Young measures we have that

$$g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n}) \overset{*}{\rightharpoonup} \int g_{k,1}(u,v)\, d\nu_{t,x} \text{ in } L^\infty \text{ for } k \in \{i,j\},$$
$$g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n}) \overset{*}{\rightharpoonup} \int g_{k,2}(u,v)\, d\nu_{t,x} \text{ in } L^\infty \text{ for } k \in \{i,j\}.$$

Since $(u^{\epsilon_n}, v^{\epsilon_n})$ is bounded, it is clear that $g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n})$ and $g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n})$ are in L^2_{loc} and hence it follows that

$$g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n}) \rightharpoonup \int g_{k,1}(u,v)\, d\nu_{t,x} \text{ in } L^2_{loc} \text{ for } k \in \{i,j\},$$
$$g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n}) \rightharpoonup \int g_{k,2}(u,v)\, d\nu_{t,x} \text{ in } L^2_{loc} \text{ for } k \in \{i,j\}.$$

Applying the Div-Curl Lemma (see Theorem 5.2.1 in [Ev 90]) to $(g_{j,1}(u^\epsilon, v^\epsilon), g_{j,2}(u^\epsilon, v^\epsilon))$ and $(-g_{i,2}(u^\epsilon, v^\epsilon), g_{i,1}(u^\epsilon, v^\epsilon))$ we have that

$$\begin{aligned} & g_{j,2}(u^{\epsilon_n}, v^{\epsilon_n})\, g_{i,1}(u^{\epsilon_n}, v^{\epsilon_n}) - g_{i,2}(u^{\epsilon_n}, v^{\epsilon_n})\, g_{j,1}(u^{\epsilon_n}, v^{\epsilon_n}) \\ & \to \left(\int_{\mathbb{R}^2} g_{j,2}(u,v)\, d\nu_{t,x}\right)\left(\int_{\mathbb{R}^2} g_{i,1}(u,v)\, d\nu_{t,x}\right) - \left(\int_{\mathbb{R}^2} g_{i,2}(u,v)\, d\nu_{t,x}\right)\left(\int_{\mathbb{R}^2} g_{j,1}(u,v)\, d\nu_{t,x}\right) \end{aligned}$$

in the sense of distributions. Hence by the fundamental theorem of Young measures, for a.e. $(t,x) \in \mathbb{R}^2$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(g_{j,2}(u,v)\, g_{i,1}(u,v) - g_{i,2}(u,v)\, g_{j,1}(u,v)\right) d\nu_{t,x} \\ & = \int_{\mathbb{R}^2} g_{j,2}(u,v)\, d\nu_{t,x} \int_{\mathbb{R}^2} g_{i,1}(u,v)\, d\nu_{t,x} - \int_{\mathbb{R}^2} g_{i,2}(u,v)\, d\nu_{t,x} \int_{\mathbb{R}^2} g_{j,1}(u,v)\, d\nu_{t,x}. \end{aligned}$$

Note that by (21), (143), (144), (13) and (14) we have that

$$P_2(u,v) = \begin{pmatrix} -g_{1,2}(u,v) & g_{1,1}(u,v) \\ -g_{2,2}(u,v) & g_{2,1}(u,v) \\ -g_{3,2}(u,v) & g_{3,1}(u,v) \\ -g_{4,2}(u,v) & g_{4,1}(u,v) \end{pmatrix}.$$

Now as

$$g_{j,2}(u,v)\, g_{i,1}(u,v) - g_{i,2}(u,v)\, g_{j,1}(u,v) = M_{ij}(P_2(u,v)) \text{ for any } 1 \le i < j \le 4$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2} g_{j,2}(u,v)\, d\nu_{t,x} \int_{\mathbb{R}^2} g_{i,1}(u,v)\, d\nu_{t,x} - \int_{\mathbb{R}^2} g_{i,2}(u,v)\, d\nu_{t,x} \int_{\mathbb{R}^2} g_{j,1}(u,v)\, d\nu_{t,x} \\ & = M_{ij}\left(\int P_2(u,v) d\nu_{t,x}\right) \text{ for any } 1 \le i < j \le 4, \end{aligned}$$

it follows that for a.e. $(t,x) \in \mathbb{R}^2$, we have $(P_2)_\sharp \nu_{t,x} \in \mathcal{M}^{pc}(\mathcal{K}_2)$. Since $\{(u^\epsilon, v^\epsilon)\}$ has small oscillation, we know that Spt$\nu_{t,x}$ is contained in a small neighborhood. It follows from Theorem 6 that $\nu_{t,x}$ is a Dirac measure for a.e. $(t,x) \in \mathbb{R}^2$. Hence there exists a subsequence $(u^{\epsilon_n}, v^{\epsilon_n})$ that converges to (u,v) strongly. □

9. Appendix: Auxiliary lemma for entropies

Let the functions a and F be as in Subsection 1.1. Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$G\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -a(y_2) \\ -y_1 \end{pmatrix},$$

and consider the system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial}{\partial x} G\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{145}$$

Define $\Phi, \Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1^2}{2} + F(y_2) \\ y_1 y_2 \end{pmatrix} \text{ and } \Psi\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 a(y_2) \\ -\left(\frac{y_1^2}{2} + a(y_2)y_2 - F(y_2)\right) \end{pmatrix}.$$

Further define η_1, q_1 and η_2, q_2 in the following way:

$$\eta_1(y_1,y_2) = \frac{y_1^2}{2} + F(y_2), \quad q_1(y_1,y_2) = -y_1 a(y_2) \tag{146}$$

and

$$\eta_2(y_1,y_2) = y_1 y_2, \quad q_2(y_1,y_2) = -\frac{y_1^2}{2} - a(y_2)y_2 + F(y_2). \tag{147}$$

Lastly, for all $\alpha \in \mathbb{R}^2$, define

$$Q_\alpha \eta_i(y_1,y_2) := \eta_i(y_1,y_2) - \eta_i(\alpha_1,\alpha_2) - \nabla\eta_i(\alpha_1,\alpha_2)\cdot(y_1-\alpha_1, y_2-\alpha_2) \tag{148}$$

and

$$Q^*_\alpha q_i(y_1,y_2) := q_i(y_1,y_2) - q_i(\alpha_1,\alpha_2) - \nabla\eta_i(\alpha_1,\alpha_2)\cdot(-a(y_2)+a(\alpha_2), -y_1+\alpha_1). \tag{149}$$

The main purpose of the following lemma is to show that the entropy/entropy flux pairs $(\eta_i, q_i), i = 1,2$, defined in (146)-(147) are the correct ones associated to the system (145). More precisely, we prove

Lemma 23. *The pair (Φ, Ψ) is an entropy/entropy flux pair for the system given by (145), i.e., (Φ, Ψ) satisfies*

$$D\Phi(y_1,y_2)DG(y_1,y_2) = D\Psi(y_1,y_2). \tag{150}$$

Further, we have that

$$Q_\alpha \eta_1(y_1,y_2) = \frac{(y_1-\alpha_1)^2}{2} + F(y_2) - F(\alpha_2) - a(\alpha_2)(y_2-\alpha_2), \tag{151}$$

$$Q^*_\alpha q_1(y_1,y_2) = -(y_1-\alpha_1)\left(a(y_2)-a(\alpha_2)\right), \tag{152}$$

$$Q_\alpha \eta_2(y_1,y_2) = (y_1-\alpha_1)(y_2-\alpha_2), \tag{153}$$

and

$$Q^*_\alpha q_2(y_1,y_2) = -\left(\frac{(y_1-\alpha_1)^2}{2} + F(\alpha_2) - F(y_2) - a(y_2)(\alpha_2-y_2)\right). \tag{154}$$

Hence (η_1, q_1) and (η_2, q_2) are the entropy/entropy flux pairs that give Equations (9.8) of DiPerna [DP 85].

Proof. Note that there is a discrepancy between the entropy/entropy flux pairs (η_1, q_1) and (η_2, q_2) defined in (146)-(147) and those given by DiPerna [DP 85]. Part of the discrepancy may be down to what DiPerna means by Legendre transform.

We can find an explicit formula for the Legendre transform in the following way. Given a convex function f, for each p we want to find the x value that maximizes $q(x) := xp - f(x)$. So we need $q'(x) = p - f'(x) = 0$, and thus $p = f'(x) =: g(x)$. Since f is convex, f' is monotonic increasing and therefore $x = (f')^{-1}(p) = g^{-1}(p)$. So the Legendre transform of f, denoted by Lf, is given by

$$Lf(p) = pg^{-1}(p) - f\left(g^{-1}(p)\right).$$

In the context of entropies in [DP 85], DiPerna defines $\tilde{\tau} := LF$ and thus $\tilde{\tau}(p) = pa^{-1}(p) - F\left(a^{-1}(p)\right)$. Further he defines

$$\tilde{q}_2(y_1, y_2) := \frac{y_1^2}{2} + \tilde{\tau}(y_2).$$

However, with $\tilde{q}_2$ defined as above along with η_2 given in (147), the pair $(\eta_2, \tilde{q}_2)$ does not form an entropy/entropy flux pair or satisfy (153) and (154), which are Equations (9.8) of DiPerna [DP 85]. However, if we define the "Legendre transform" to be

$$\tau(p) := \tau\left(a(p)\right) = a(p)p - F(p), \tag{155}$$

then up to a sign we obtain the definition of entropies given by DiPerna (see Equation (1.11) in [DP 85]), because with the definition of τ, we have

$$q_2(y_1, y_2) \overset{(147),(155)}{=} -\frac{y_1^2}{2} - \tau(y_2).$$

Note that

$$\frac{d}{dp}\left(\tau(p)\right) \overset{(155)}{=} \frac{d}{dp}\left(a(p)p - F(p)\right) = a'(p)p + a(p) - a(p) = a'(p)p. \tag{156}$$

Step 1. We establish (150) to show that $(\eta_i, q_i), i = 1, 2$, are entropy/entropy flux pairs associated to the system (145).

Proof of Step 1. We calculate

$$D\Phi(y_1, y_2) = \begin{pmatrix} y_1 & a(y_2) \\ y_2 & y_1 \end{pmatrix},$$

$$D\Psi(y_1, y_2) \overset{(156)}{=} \begin{pmatrix} -a(y_2) & -y_1 a'(y_2) \\ -y_1 & -y_2 a'(y_2) \end{pmatrix}$$

and

$$DG(y_1, y_2) = \begin{pmatrix} 0 & -a'(y_2) \\ -1 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} D\Phi(y_1, y_2) DG(y_1, y_2) &= \begin{pmatrix} y_1 & a(y_2) \\ y_2 & y_1 \end{pmatrix} \begin{pmatrix} 0 & -a'(y_2) \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -a(y_2) & -a'(y_2)y_1 \\ -y_1 & -a'(y_2)y_2 \end{pmatrix} = D\Psi(y_1, y_2). \end{aligned}$$

This establishes (150).

Step 2. We establish (151)-(154) to show that $(\eta_i, q_i), i = 1, 2$, give Equations (9.8) of [DP 85].

Proof of Step 2. First note that $\nabla\eta_1(\alpha_1,\alpha_2) = (\alpha_1, a(\alpha_2))$. We calculate

$$\begin{aligned} Q_\alpha\eta_1(y_1,y_2) &\overset{(148),(146)}{=} \frac{y_1^2}{2} + F(y_2) - \frac{\alpha_1^2}{2} - F(\alpha_2) - (\alpha_1, a(\alpha_2)) \cdot (y_1-\alpha_1, y_2-\alpha_2) \\ &= \frac{y_1^2}{2} + F(y_2) - \frac{\alpha_1^2}{2} - F(\alpha_2) - \alpha_1(y_1-\alpha_1) - a(\alpha_2)(y_2-\alpha_2) \\ &= \frac{(y_1-\alpha_1)^2}{2} + F(y_2) - F(\alpha_2) - a(\alpha_2)(y_2-\alpha_2), \end{aligned}$$

and hence we have (151). Further we have

$$\begin{aligned} Q_\alpha^* q_1(y_1,y_2) &\overset{(149),(146)}{=} -y_1 a(y_2) + \alpha_1 a(\alpha_2) - (\alpha_1, a(\alpha_2)) \cdot (-a(y_2)+a(\alpha_2), -y_1+\alpha_1) \\ &= -y_1 a(y_2) + \alpha_1 a(\alpha_2) + \alpha_1 a(y_2) - \alpha_1 a(\alpha_2) + a(\alpha_2) y_1 - a(\alpha_2)\alpha_1 \\ &= -(y_1-\alpha_1)\left(a(y_2)-a(\alpha_2)\right), \end{aligned}$$

and thus we have (152). Next note that $\nabla\eta_2(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1)$, and we have

$$\begin{aligned} Q_\alpha\eta_2(y_1,y_2) &\overset{(147),(148)}{=} y_1y_2 - \alpha_1\alpha_2 - (\alpha_2,\alpha_1)\cdot(y_1-\alpha_1, y_2-\alpha_2) \\ &= y_1y_2 - \alpha_1\alpha_2 - \alpha_2 y_1 + \alpha_2\alpha_1 - \alpha_1 y_2 + \alpha_1\alpha_2 \\ &= (y_1-\alpha_1)(y_2-\alpha_2), \end{aligned}$$

which gives (153). Finally we have

$$\begin{aligned} Q_\alpha^* q_2(y_1,y_2) &\overset{(147),(149)}{=} -\frac{y_1^2}{2} - y_2 a(y_2) + F(y_2) + \frac{\alpha_1^2}{2} + \alpha_2 a(\alpha_2) \\ &\qquad - F(\alpha_2) - (\alpha_2,\alpha_1)\cdot(-a(y_2)+a(\alpha_2), -y_1+\alpha_1) \\ &= -\frac{y_1^2}{2} - y_2 a(y_2) + F(y_2) + \frac{\alpha_1^2}{2} + \alpha_2 a(\alpha_2) - F(\alpha_2) + \alpha_2 a(y_2) \\ &\qquad - \alpha_2 a(\alpha_2) + \alpha_1 y_1 - \alpha_1^2 \\ &= -\left(\frac{(y_1-\alpha_1)^2}{2} + F(\alpha_2) - F(y_2) - a(y_2)(\alpha_2-y_2)\right), \end{aligned}$$

and thus we have (154). □

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