Hamilton Jacobi Isaacs equations for Differential Games with asymmetric information on probabilistic initial condition.

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Abstract

We investigate Hamilton Jacobi Isaacs equations associated to a two-players zero-sum differential game with incomplete information. The first player has complete information on the initial state of the game while the second player has only information of a - possibly uncountable - probabilistic nature: he knows a probability measure on the initial state. Such differential games with finite type incomplete information can be viewed as a generalization of the famous Aumann-Maschler theory for repeated games. The main goal and novelty of the present work consists in obtaining and investigating a Hamilton Jacobi Isaacs Equation satisfied by the upper and the lower values of the game. Since we obtain a uniqueness result for such Hamilton Jacobi equation, as a byproduct, this gives an alternative proof of the existence of a value of the differential game (which has been already obtained in the literature by different technics). Since the Hamilton Jacobi equat ion is natural ly stated in the space of probability measures, we use the Wasserstein distance and some tools of optimal transport theory.

Key words. Differential game; asymmetric information; Isaacs condition; continuous initial distribution; Wasserstein distance; Functional on measures.

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Introduction

In this paper we study a zero-sum two-players differential game where the first player has asymetric information on the initial position. The dynamics is given by

$$x'(t) = f(x(t), u(t), v(t)),\ \ u(t) \in U,\ v(t) \in V \tag{1}$$

with $f : \mathbb{R}^N \times U \times V$ and where U and V are compact subsets of some finite dimensional spaces. A payoff is given by a function $g : \mathbb{R}^N \mapsto \mathbb{R}$.

The first player acts on the system (1) by choosing a measurable control $u : [0, T] \mapsto U$, he tries to minimize a final cost $g(x(T))$. The second player wishes to maximize the final cost $g(x(T))$ by acting on the dynamics through the choice of a measurable control $v : [0, T] \mapsto V$.

Later on we will make suitable suppositions such that as soon as the initial position x_0 and the measurable controls $u(\cdot)$ and $v(\cdot)$ are known, there exists a unique solution to (1) satisfying $x(t_0) = x_0$.

Let us now describe how the game is played. Here $X \subset \mathbb{R}^N$ is a given compact subset invariant for the dynamics (1).

- before the game starts, the initial position $x_0 \in X$ is chosen randomly according to a probability measure μ_0 which support is included in X,
- the initial state x_0 is communicated to Player I but not to Player II,
- the game is played on the time interval $[t_0, T]$,
- both players know the probability μ_0 and observe their opponent's controls during the game.

Such game has been investigated in the framework of repeated (discrete time) games in [5, 24, 25]. For differential games the corresponding problem has been studied in [10, 11, 28] in the case where the asymetric information is of finite type (in our framework this means that the probability measure μ_0 has a finite support).

It is worth pointing out that the role of the information is crucial in the game studied here. Indeed Player II does not know what is the current state of the game. He can only try to guess it by observing the actions of his opponent. Knowing this, the first player wishes to hide his actions by playing randomly. This leads to a specific context of strategies [4, 8, 9, 29].

Following the reasoning of [19] for full information differential games, the existence of the value is obtained by showing that the upper value and the lower value both satisfies a partial differential equation (Hamilton Jacobi Isaacs equation) which has a unique solution. In the case of a measure μ_0 with finite support - as it is studied in [10, 11] the Hamilton Jacobi Bellman equation could be considered in a finite dimensional space of dimension NI (where I is the cardinal of the support of μ_0) where finite dimensional pde analysis can be done (with a specific notion of dual viscosity solutions).

In the general case where the information is not of finite type - namely the support of μ_0 is not finite - the analysis is much more difficult. A first approach consists in proving that the value function is continuous with respect to μ_0 and to approximate μ_0 by a sequence of probability measures with finite support. This allows to prove the existence of a value [14, 22]. However this approach does not give any information on the Hamilton Jacobi equation that should be satisfied by the value.

The main goal of the present article is to obtain and study a Hamilton Jacobi Isaacs Equation satisfied by the value. The main difficulty lies in the fact that the value depends on a probability measure. So we will give a meaning of a pde of Hamilton Jacobi type on the Wasserstein space of probability measures. To accomplish this task we were inspired by [9] (cf also [13]) and we define a notion of dual viscosity solution for Hamilton Jacobi Isaacs equations associated with incomplete information differential games. We also prove a uniqueness result of such pde, this will give as a byproduct the existence of the value. In the literature several other notions of solution of partial differential equation in the

Wasserstein space have been studied in different context (for instance cf [2, 3, 15, 20] among many others and the huge literature on mean field games

cf e.g. [23]). In our work we need a very precise notion of dual viscosity solution, well adapted to differential games, which can be viewed as an extension of the viscosity solutions introduced in [9].

Let us describe how the paper is organized: The first section contains the description of the game together with some basic facts on the Wasserstein space of measures and on the Isaacs' condition. Section 2 concerns the definition and the study of suitable "extended values". The third section is devoted to appropriate dynamical principle properties. In section 4, a notion of viscosity solution for Hamilton Jacobi Isaacs equation is introduced and a comparison theorem is provided (with a uniqueness result as its consequence). In the last section, the extended values are shown to satisfy the Hamilton Jacobi equation and the existence of the value of the game is derived.

1 Preliminaries and Assumptions

Throughout the paper, finite dimensional spaces are equipped with the euclidean norm denoted $|x|$ associated with the scalar product denoted by $x.y$, the closed ball of center x and of radius $r > 0$ is denoted by $B(x, r)$ while B stands for the closed unit ball. The Lebesgue measure on $\mathbb{R}^N$ is denoted by $\mathcal{L}^N$. The notation $\mathcal{C}(X, Y)$ stands for the set of continuous functions from the space X to Y while $\mathcal{C}(X)$ is the set of continuous functions from X to $\mathbb{R}$.

1.1 Dynamics

The set $\mathcal{U}([t_0, T])$ denotes the set of all measurable controls from $[t_0, T]$ to U. When there is no ambiguity we shorten this notation in $\mathcal{U}(t_0)$ or $\mathcal{U}$. Similarly the set of measurable controls from $[t_0, T]$ to V is denoted by $\mathcal{V}([t_0, T])$ (in short $\mathcal{V}(t_0)$ or $\mathcal{V}$) .

The function $f : \mathbb{R}^N \times U \times V$ which appears in the dynamics (1) is assumed to be continuous with respect to all variables and Lipschitz continuous in the first variable uniformly with respect to (u, v). Then, it is well-known that for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$, associated with the initial condition $x(t_0) = x_0$ there is a unique absolutely continuous solution to (1) denoted by $t \mapsto X_t^{t_0,x_0,u,v}$ which is defined on $[t_0, +\infty[$. Standard estimates show that there exists a constant $C > 0$ such that for all $x, x' \in \mathbf{R}^n$ and all $s \in [t_0, T]$,

$$\left|X_s^{t_0,x,u,v} - x\right| \leq C|s - t_0| \tag{2}$$

$$\left|X_s^{t_0,x,u,v} - X_s^{t_0,x',u,v}\right| \leq C\,|x - x'|\,. \tag{3}$$

Throughout the paper we will make also the following assumption:

$$\text{There exists } X \subset \mathbb{R}^N \text{ a compact invariant set for (1).} \tag{4}$$

The fact that the set X is invariant means that for any $(t_0, x_0) \in [0, +\infty) \times X$, for all $(u, v) \in \mathcal{U} \times \mathcal{V}$ the associated solution remains forever in X : $X_t^{t_0,x_0,u,v} \in X$ for any $t \geq t_0$.

The cost function $g : \mathbb{R}^N \mapsto \mathbb{R}$ is supposed to be bounded and Lipschitz continuous.

1.2 Probability Measures on the initial conditions

The notation $\Delta(X)$ stands for the set of Borel probability measures on $X \subset \mathbb{R}^N$. For any $p \in [1, +\infty)$ one can define the Wasserstein distance between the measures $\mu \in \Delta(X)$ and $\nu \in \Delta(X)$ as follows

$$W_p(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \left(\int_{X^2} |x-y|^p d\gamma(x,y) \right)^{\frac{1}{p}} \right\}$$

where $\Pi(\mu, \nu)$ is the set of probability measures γ on X^2 which has μ as first marginal and ν as second one. It is known that there exists an optimal measure γ achieving the above infimum (such a γ is called an optimal plan from μ to ν). It is also well known that when X is compact, the distance W_p is compatible with the weak star convergence of measures, that is for any $(\mu_n)_n$ and μ in $\Delta(X)$:

$$\lim_{n\to+\infty} W_p(\mu_n, \mu) = 0 \Leftrightarrow \lim_{n\to+\infty} \int_X \varphi(x)\ d\mu_n(x) = \int_X \varphi(x)\ d\mu(x) \quad \forall \varphi \in \mathcal{C}(X).$$

We refer the reader to [30] (Theorem 7.12 p 212) or [27] (section 5.2. p183) for such basic facts on Wasserstein distance and optimal transport. For $\mu \in \Delta(\mathbb{R}^N)$ and $\phi : \mathbb{R}^N \to \mathbb{R}^N$ a Borel measurable bounded function, we denote by $\phi \sharp \mu$ the push-forward of μ by ϕ, namely the measure in $\Delta(\mathbb{R}^N)$ such that

$$\phi \sharp \mu(A) = \mu\left(\phi^{-1}(A)\right) \qquad \text{for any Borel set } A \subset \mathbb{R}^N.$$

We will consider the following duality pairing:

$$(\mu, \varphi) \in \mathcal{M}_b(X) \times \mathcal{C}(X) \mapsto \langle \mu, \varphi \rangle = \int_X \varphi(x)\ d\mu(x)$$

where $\mathcal{M}_b(X)$ stands for the space of bounded Borel measures on X. Then for any $z : \mathcal{C}(X) \to \mathbb{R}$, we recall the definitions of the Fenchel conjugate and biconjugate of z :

$$z^*(\mu_0) = \sup_{\varphi \in \mathcal{C}(X)} \{\langle \mu_0, \varphi \rangle - z(\varphi)\}, \quad z^{**}(\varphi_0) = \sup_{\mu \in \mathcal{M}_b(X)} \{\langle \mu, \varphi_0 \rangle - z^*(\mu)\}.$$

The key point is that if z is convex and lower semi-continuous for the uniform topology, $z = z^{**}$ (see for instance [6], p95). In the same way, any $V : \Delta(X) \to R$, can be extended to $\mathcal{M}_b(X)$ by setting $V \equiv +\infty$ outside $\Delta(X)$, this allows to define its Fenchel conjugate and its convex subdifferential:

$$V^*(\varphi) = \sup_{\mu \in \Delta(X)} \{\langle \mu, \varphi \rangle - V(\mu)\}, \quad \forall \varphi \in \mathcal{C}(X),$$

$$\partial_- V(\mu) = \{\varphi \in \mathcal{C}(X) :\ V(\nu) \geq V(\mu) + \langle \nu - \mu, \varphi \rangle, \quad \forall \nu \in \Delta(X)\}.$$

We will also use the following notation:

$$\langle \Phi, \Psi \rangle_{L^2_\mu} = \int_X \Phi(x) \cdot \Psi(x) d\mu(x) \quad \forall \mu \in \Delta(X),\ \Phi,\ \Psi \in \mathcal{C}(X, X).$$

1.3 The Isaacs' condition

In differential games, the Isaacs' condition is a natural assumption for proving the existence of the value [19], without this condition the study is much more complex [22] if it would be tractable. In our framework the Isaacs' condition takes the following form

$$\forall(\mu,p)\in\Delta(X)\times C(X,\mathbb{R}^N),$$
$$\inf_{u\in U}\sup_{v\in V}\int_X f(x,u,v)\cdot p(x)\ d\mu(x)=\sup_{v\in V}\inf_{u\in U}\int_X f(x,u,v)\cdot p(x)\ d\mu(x). \tag{5}$$

When μ reduce to the Dirac measure δ_x and p is a constant function, the equation (5) becomes the usual Isaacs' condition for differential games with full information [8].

It is useful to have several equivalent formulations of the Isaacs' condition.

Proposition 1 *The conditions below are both equivalent to the Isaacs' condition (5):*

(I1) For all $n\in\mathbb{N}$, $\mu=\sum_{i=1}^n c_i\delta_{x_i}\in\Delta(X)$ *and* $p_1,\dots,p_n\in\mathbb{R}^N$*:*

$$\inf_{u\in U}\sup_{v\in V}\sum_{i=1}^n c_i\ f(x_i,u,v)\cdot p_i=\sup_{v\in V}\inf_{u\in U}\sum_{i=1}^n c_i\ f(x_i,u,v)\cdot p_i.$$

(I2) For all $(\mu,p,\Phi)\in\Delta(X)\times\mathcal{C}(X,\mathbb{R}^N)\times\mathcal{C}(X,X)$*:*

$$\inf_{u\in U}\sup_{v\in V}\int_X f(\Phi(x),u,v)\cdot p(x)\ d\mu(x)=\sup_{v\in V}\inf_{u\in U}\int_X f(\Phi(x),u,v)\cdot p(x)\ d\mu(x).$$

Proof: One can easily deduce $(I2)\Rightarrow(5)\Rightarrow(I1)$.

Let us first obtain a preliminary estimate. Fix $\Phi\in\mathcal{C}(X,X)$, $p\in\mathcal{C}(X,\mathbb{R}^N)$, $\mu,\nu\in\Delta(X)$. Let $\gamma\in\Delta(X\times X)$ be an optimal transport map for $W_2(\mu,\nu)$. It holds:

$$\begin{aligned}&\int_X f(\Phi(x),u,v)\cdot p(x)\ d\mu(x)-\int_X f(\Phi(x),u,v)\cdot p(x)\ d\nu(x)\\ \le\ & Lip(f)\|p\|_\infty\left(\int_{X\times X}|\Phi(x)-\Phi(y)|^2\ d\gamma(x,y)\right)^{1/2}+\|f\|_\infty\left(\int_{X\times X}|p(x)-p(y)|^2\ d\gamma(x,y)\right)^{1/2}.\end{aligned}$$

Indeed:

$$\begin{aligned}&\int_X f(\Phi(x),u,v)\cdot p(x)\ d\mu(x)-\int_X f(\Phi(x),u,v)\cdot p(x)\ d\nu(x)\\ &=\int_{X\times X} f(\Phi(x),u,v)\cdot p(x)-f(\Phi(y),u,v)\cdot p(y)\ d\gamma(x,y)\\ &\le\int_{X\times X} f(\Phi(x),u,v)\cdot(p(x)-p(y))\ d\gamma(x,y)+\int_{X\times X}(f(\Phi(x),u,v)-f(\Phi(y),u,v))\cdot p(y)\ d\gamma(x,y).\end{aligned}$$

Now we show $(I1)\Rightarrow(5)$. Fix $(\mu,p)\in\Delta(X)\times\mathcal{C}(X,\mathbb{R}^N)$ and $(\mu_n)_n$ with finite support converging to μ for the weak star topology. Denoting by (γ_n) the optimal map for $W_2(\mu,\mu_n)$ and assuming $(I1)$, it holds:

$$\begin{aligned}
&\inf_{u\in U}\sup_{v\in V}\int_X f(x,u,v)\cdot p(x)\ d\mu(x) \le \inf_{u\in U}\sup_{v\in V}\int_X f(x,u,v)\cdot p(x)\ d\mu_n(x) + Lip(f)\|p\|_\infty W_2(\mu_n,\mu)\\
&\qquad +\|f\|_\infty\left(\int_{X\times X}|p(x)-p(y)|^2\ d\gamma_n(x,y)\right)^{1/2}\\
=&\ \sup_{v\in V}\inf_{u\in U}\int_X f(x,u,v)\cdot p(x)\ d\mu_n(x) + Lip(f)\|p\|_\infty W_2(\mu_n,\mu)\\
&\qquad +\|f\|_\infty\left(\int_{X\times X}|p(x)-p(y)|^2\ d\gamma_n(x,y)\right)^{1/2}\\
\le&\ \sup_{v\in V}\inf_{u\in U}\int_X f(x,u,v)\cdot p(x)\ d\mu(x) + 2Lip(f)\|p\|_\infty W_2(\mu_n,\mu)\\
&\qquad +2\|f\|_\infty\left(\int_{X\times X}|p(x)-p(y)|^2\ d\gamma_n(x,y)\right)^{1/2}.
\end{aligned}$$

By Prokhorov's theorem, up to a subsequence, γ_n tends to the map $\gamma\in\Delta(X\times X)$, which is furthermore optimal for $W_2(\mu,\mu)$ and is defined by

$$\int_{X\times X}\varphi(x,y)\ d\gamma(x,y)=\int_X\varphi(x,x)\ d\mu(x)\quad \forall\varphi\in\mathcal{C}(X).$$

Then the previous inequality implies

$$\inf_{u\in U}\sup_{v\in V}\int_X f(x,u,v)\cdot p(x)\ d\mu(x)\le \sup_{v\in V}\inf_{u\in U}\int_X f(x,u,v)\cdot p(x)\ d\mu(x).$$

Because the reverse inequality is trivial, this provides the desired result $(I_1)\Rightarrow(5)$.

It remains to prove $(I1)\Rightarrow(I2)$. Suppose (I1). It is enough to show (I2) for probability measures with finite supports. Let $\Phi\in\mathcal{C}(X,X)$, $p\in\mathcal{C}(X,\mathbb{R}^N)$ and $\mu=\sum_{i=1}^n c_i\delta_{x_i}\in\Delta(X)$. We rewrite μ as

$$\mu=\sum_{i\in I}\ \sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j\ \delta_{x_i}$$

$$\text{with } I:=\{i\in\{1,\dots,n\}:\ i=\min\{j:\ \Phi(x_i)=\Phi(x_j)\}\}.$$

Then:

$$\begin{aligned}
&\inf_{u\in U}\sup_{v\in V}\int_X f(\Phi(x),u,v)\cdot p(x)\ d\mu(x)=\inf_{u\in U}\sup_{v\in V}\sum_{i=1}^n c_i f(\Phi(x_i),u,v)\cdot p(x_i)\\
=&\ \inf_{u\in U}\sup_{v\in V}\sum_{i\in I}\left(\sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j p(x_j)\right)\cdot f(\Phi(x_i),u,v)\\
=&\ \inf_{u\in U}\sup_{v\in V}\sum_{i\in I}\left(\sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j\right)\ f(\Phi(x_i),u,v)\cdot\left(\frac{\sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j p(x_j)}{\sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j}\right).
\end{aligned}$$

Applying (I1) with the probability measure $\sum_{i\in I}\left(\sum_{j:\ \Phi(x_j)=\Phi(x_i)} c_j\right)\delta_{\Phi(x_i)}$ gives the desired result. The proof is complete.

QED

1.4 Strategies and Values

Now we define values and strategies of the game. Because of the asymmetric structure of information, the strategies of the players should involve only their available information. This leads to the following notion of strategies (comp. [10, 11, 12, 14]).

Definition 1 *Let S be the set of triples $(\Omega,\mathcal{F},P)$ such that $\Omega=[0,1]^m$ for some m, $\mathcal{F}$ is a σ-field contained in the class of Borel sets $B([0,1]^m)$ and P a probability measure on $(\Omega,\mathcal{F})$.*

A random strategy for Player II is a pair $((\Omega_\beta,\mathcal{F}_\beta,P_\beta),\beta)$ where $\beta:\Omega_\beta\times\mathcal{U}(t_0)\to\mathcal{V}(t_0)$ is a Borel measurable[1] map and there exists a delay $\tau_\beta>0$ such that for all $\omega_\beta\in\Omega_\beta$ $\beta(\omega_\beta,\cdot):\mathcal{U}(t_0)\mapsto\mathcal{V}(t_0)$ is nonanticipative with delay τ_β. Namely there exists $\tau_\beta>0$ such that for any $u_1,u_2\in\mathcal{U}(t_0)$, for any $t\in[t_0,T)$, if $u_1=u_2$ a.e. on $[t_0,t]$, then $\beta(\omega_\beta,u_1)=\beta(\omega_\beta,u_2)$ a.e. on $[t_0,(t+\tau_\beta)\wedge T]$.

A random strategy for Player I is a pair $((\Omega_\alpha,\mathcal{F}_\alpha,P_\alpha),\alpha)$ where $(\Omega_\alpha,\mathcal{F}_\alpha,P_\alpha)\in S$, such that there exists a delay $\tau_\alpha>0$ with

1- the map $\alpha:X\times\Omega_\alpha\times\mathcal{V}(t_0)\to\mathcal{U}(t_0)$ is Borel measurable,

2- for any $\omega_\alpha\in\Omega_\alpha$ and $x\in\mathbb{R}^N$, the strategy $v\in\mathcal{V}(t_0)\mapsto\alpha(x,\omega_\alpha,v)$ is non anticipative with delay τ_α.

Sets of random strategies for Players I and II are denoted by $A_r(t_0)$ and $B_r(t_0)$.

Now we associate to any pair of random strategies a trajectory thanks to the Lemma below. This enables us to write the game in a normal form.

Lemma 1 *([14]) For any $(\alpha,\beta)\in A_r(t_0)\times B_r(t_0)$, for any $\omega:=(\omega_\alpha,\omega_\beta)\in\Omega_\alpha\times\Omega_\beta$ and for any initial condition x_0, there is a unique pair $(u_{\omega,x_0},v_{\omega,x_0})\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$, such that*

$$\alpha(x_0,\omega_\alpha,v_{\omega,x_0})=u_{\omega,x_0}\text{ and }\beta(\omega_\beta,u_{\omega,x_0})=v_{\omega,x_0}\ . \tag{6}$$

Furthermore the map $(\omega,x_0)\mapsto(u_{\omega,x_0},v_{\omega,x_0})\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ is Borel measurable.

Consequently to $(\alpha,\beta)\in A_r(t_0)\times B_r(t_0)$ we may associate a trajectory defined by

$$\forall t\geq t_0,\quad X_t^{t_0,x_0,\alpha(x,\omega_\alpha,\cdot),\beta(\omega_\beta,\cdot)}=X_t^{t_0,x_0,u_{\omega,x_0},v_{\omega,x_0}}$$

where u_{ω,x_0} and v_{ω,x_0} are associated to (α,β) by the Lemma 1.

[1] This means that the measurability property is considered when $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the Borel σ-field associated with $L^1_U[t_0,T]$ and $L^1_V[t_0,T]$.

Definition 2 *(Values in Random Strategies) Fix $t_0 \in [0,T]$ and $\mu_0 \in \Delta(X)$. To any $(\alpha,\beta) \in A_r(t_0) \times B_r(t_0)$ we associate the following cost*

$$J(t_0,\mu_0,\alpha,\beta) := \int_{\Omega_\alpha}\int_{\Omega_\beta}\int_X g(X_T^{t_0,x,\alpha(x,\omega_\alpha,\cdot)\beta(\omega_\beta,\cdot)})\, d\mu_0(x)\, dP_\alpha(\omega_\alpha)\, dP_\beta(\omega_\beta),$$

which enables us to define the upper and lower values of the game as follows

$$V_r^+(t_0,\mu_0) := \inf_{\alpha\in A_r(t_0)} \sup_{\beta\in B_r(t_0)} J(t_0,\mu_0,\alpha,\beta), \tag{7}$$

$$V_r^-(t_0,\mu_0) := \sup_{\beta\in B_r(t_0)} \inf_{\alpha\in A_r(t_0)} J(t_0,\mu_0,\alpha,\beta). \tag{8}$$

We recall that we have proved in [14] that for any $t_0 \in [0,T]$, the values with random strategies $V_r^-(t_0,\cdot)$ and $V_r^+(t_0,\cdot) : \Delta(X) \mapsto \mathbb{R}$ are Lipschitz continuous with respect to the Wasserstein distance W_2. Moreover assuming Isaacs' condition (5) both values coincide namely $V_r^- = V_r^+$.

It is also possible to define in a similar way the notion of pure strategies as follows

Definition 3 *A pure strategy for Player II is a Borel measurable map $\beta : \mathcal{U}(t_0) \mapsto \mathcal{V}(t_0)$ which is nonanticipative with delay (NAD in short).*

A pure strategy for Player I is a Borel measurable map: $\alpha : \mathbb{R}^N \times \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ for which there is a delay $\tau_\alpha > 0$ such that, for any $x_0 \in \mathbb{R}^N$, the map $\alpha(x_0,\cdot) : \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ is nonanticipative with delay τ_α .
The set of pure strategies for Player I (resp. Player II) is denoted by $A(t_0)$ (resp. $B(t_0)$).

Then it is almost classical [10, 11, 14] in differential games that for obtaining the value one can play a random strategy against a pure strategy:

$$V_r^+(t_0,\mu_0) = \inf_{\alpha\in A_r(t_0)} \sup_{\beta\in B(t_0)} J(t_0,\mu_0,\alpha,\beta),\ V_r^-(t_0,\mu_0) = \sup_{\beta\in B_r(t_0)} \inf_{\alpha\in A(t_0)} J(t_0,\mu_0,\alpha,\beta). \tag{9}$$

In a way similar to Lemma 1, to a pair of pure strategies $(\alpha,\beta) \in A(t_0) \times B(t_0)$ is associated a trajectory $X_t^{t_0,x_0,\alpha(x,\cdot),\beta(\cdot)}$. So to (t_0,μ_0) one can associate a cost again denoted $J(t_0,\mu_0,\alpha,\beta)$ and corresponding upper and lower pure strategies values (cf [10, 14]).

In [14] we have also shown that if the probability measure $\mu_0 \in \Delta(X)$ has not any atoms, then the value function $V_r(t_0,\mu_0)$ coincide with the value function defined with pure strategy (this result is false as soon as μ_0 has an atom).

2 On Extended Values

Our goal is to study $V_r^\pm(t_0,\mu_0)$ and to prove the existence of the value. For this we shall define "extended values".

We now try to motivate the necessity to extend the value in the more simple case where μ_0 has a finite support of cardinal I. In this case $\mu_0 = \sum_{i=1}^I p_i\delta_{x_i}$ where $p = (p_1, p_2 \cdots p_I)$ is a probability on the finite set $I := \{1,2,\dots I\}$ (shortly $p \in \Delta(I)$) and $\mathbf{x} = (x_1,\dots,x_I) \in X^I$. In such a way one can easily define a function $\underline{v} : [0,+\infty) \times X^I \times \Delta(I)$ such that

$V_r(t_0,\mu_0)=\underline{v}(t_0,\mathbf{x},p)$. There is mainly two advantages in studying $\underline{v}$ instead V_r. The first one is that it is possible to write a suitable Hamilton Jacobi Isaacs equation[2] satisfied by $\underline{v}$ and this is crucial for proving the existence of a value (cf [10]). The second advantage lies in the fact that there is a nice separation between the variable $\mathbf{x}$ which concerns only the support of μ_0 and the "intensity" p of the probability μ_0..

Unfortunately such a nice decomposition is not possible for a general measure μ_0 which support is not necessarily finite (and one cannot dream to reduce to a pde in a finite dimensional space). However we now explain how to obtain a function which plays the role of $\underline{v}$ in the case of measures with finite support.

2.1 Definition of Extended Values

Definition 4 *Let $t_0\in[0,T]$, $\mu_0\in\Delta(X)$, $\Phi\in \mathrm{L}^2_{\mu_0}(X,X)$, we set:*

$$(10)\quad \mathcal{V}_r^+(t_0,\Phi,\mu_0):=\inf_{\alpha\in A_r(t_0)}\sup_{\beta\in B(t_0)}\int_{\Omega_\alpha}\int_{\mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ d\mu_0(x)dP_\alpha(\omega),$$

$$(11)\quad \mathcal{V}_r^-(t_0,\Phi,\mu_0):=\sup_{\beta\in B_r(t_0)}\inf_{\alpha\in A(t_0)}\int_{\Omega_\beta}\int_{\mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\cdot)\beta(\omega,\cdot)})\ d\mu_0(x)dP_\beta(\omega).$$

We will use the following notation:

$$\mathcal{J}(t_0,\Phi,\mu_0,\alpha,\beta):=\int_{\Omega_\alpha}\int_{\mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ d\mu_0(x)dP_\alpha(\omega).$$

We now state and prove some relation between $V_r^\pm$ and $\mathcal{V}_r^\pm$.

Lemma 2 *Consider $t\in[t_0,T]$, $\mu_0\in\Delta(X)$. Then,*

$$(12)\qquad \mathcal{V}_r^\pm(t_0,Id,\mu_0)=V_r(t_0,\mu_0),$$

$$(13)\qquad \forall\Phi\in L^2_{\mu_0}(X,X),\ \mathcal{V}_r^\pm(t_0,\Phi,\mu_0)=V_r^\pm(t_0,\Phi\sharp\mu_0).$$

Proof: The relation (12) is an obvious consequence of the definition of $\mathcal{V}_r^\pm$.
We do the proof only for $\mathcal{V}_r^+$ and we use a similar argument as in the proof of Proposition 3 in [14]. Fix $\Phi\in L^2_{\mu_0}(X,X)$. In the definition 4 of $\mathcal{V}_r^\pm(t_0,\Phi,\mu_0)$, restricting the choice of α to the $\hat\alpha$ of the form $\hat\alpha(x,\cdot)=\alpha(\Phi(x),\cdot)$ increases the value. This gives:

$$\mathcal{V}_r^\pm(t_0,\Phi,\mu_0)\le V_r^\pm(t_0,\Phi\sharp\mu_0).$$

Let us now prove the opposite inequality. Let $\varepsilon>0$ and $((\Omega_0,\mathcal{F}_0,P_0),\alpha_0)$ a strategy for player I such that:

$$(14)\qquad \sup_{\beta\in B(t_0)}\int_{\Omega_0}\int_X g(X_T^{t_0,\Phi(x),\alpha_0(x,\omega,\cdot),\beta})\ dP_0(\omega)d\mu_0(x)\le \mathcal{V}_r^+(t_0,\Phi,\mu_0)+\varepsilon.$$

State $\gamma=(Id\times\Phi)\sharp\mu_0$ and disintegrate γ with respect to $\Phi\sharp\mu_0$ as follows:

$$d\gamma(x,y)=d\gamma^y(x)\otimes d(\Phi\sharp\mu_0)(y).$$

[2]in the finite dimensional space $[0,+\infty)\times X^I\times\Delta(I)$

Take $\xi : X \times [0,1]^N \to \mathbb{R}^N$ such that for $\Phi\sharp\mu_0$-almost every $y \in X$ the function $\xi(y,\cdot)$ is an optimal transport map from $\mathcal{L}^N\lfloor[0,1]^N$ to γ^y. This means that for $\Phi\sharp\mu_0$-almost every $y \in X$ we have:

$$\xi(y,\cdot)\sharp\mathcal{L}^N\lfloor[0,1]^N = \gamma^y \ , \quad W_2^2(\mathcal{L}^N\lfloor[0,1]^N, \gamma^y) = \int_{[0,1]^N} |\omega' - \xi(y,\omega')|^2 \, d\omega'.$$

We can prove as in [14] that ξ is measurable. We now build a new strategy $((\Omega_0 \times [0,1]^N, P_0 \times \mathcal{L}^N\lfloor[0,1]^N, \mathcal{F}_0 \otimes B([0,1]^N)), \alpha_1)$ for player I by setting:

$$\alpha_1(y,\omega,\omega',\cdot) = \alpha_0(\xi(y,\omega'),\omega,\cdot).$$

Then we get

$$\begin{aligned}
V_r^+(t_0,\Phi\sharp\mu_0) &\leq \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N}\int_{\Omega_0\times[0,1]^N} g(X_T^{t_0,y,\alpha_1(y,\omega,\omega',\cdot),\beta}) \, dP_0(\omega)d\omega' d(\Phi\sharp\mu_0)(y) \\
&\leq \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N}\int_{\Omega_0\times[0,1]^N} g(X_T^{t_0,y,\alpha_0(\xi(y,\omega'),\omega,\cdot),\beta}) \, d(\Phi\sharp\mu_0)(y) dP_0(\omega)d\omega' \\
&= \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N}\int_{\Omega_0} \left[\int_{\mathbb{R}^N} g(X_T^{t_0,y,\alpha_0(x,\omega,\cdot),\beta}) \, d(\xi(y,\cdot)\sharp\mathcal{L}^N\lfloor[0,1]^N)(x)\right] d\mu_0(y)dP_0(\omega) \\
&= \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N\times\mathbb{R}^N}\int_{\Omega_0} g(X_T^{t_0,y,\alpha_0(x,\omega,\cdot),\beta}) \, d\gamma(x,y)dP_0(\omega) \\
&= \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N\times\mathbb{R}^N}\int_{\Omega_0} g(X_T^{t_0,\Phi(x),\alpha_0(x,\omega,\cdot),\beta}) \, d\mu_0(x,y)dP_0(\omega) \leq \mathcal{V}_r^+(t_0,\Phi,\mu_0) + \varepsilon,
\end{aligned}$$

the last inequality coming from (14). Making $\varepsilon \to 0$ gives the result.

QED

2.2 Regularity of the extended values

Lemma 3 *For any $t_0 \in [0,T]$ and $\Phi \in \mathrm{L}^2_{\mu_0}(X,X)$, the map $\mu_0 \mapsto \mathcal{V}_r^-(t_0,\Phi,\mu_0)$ is convex.*

The proof of this lemma is adapted from [10], lemma 3.2.
Proof: Fix $\mu_0, \mu_1 \in \Delta(X)$, $\lambda \in [0,1]$. Set $\mu_\lambda := (1-\lambda)\mu_0 + \lambda\mu_1$. Since μ_0 and μ_1 are clearly absolutely continuous with respect to μ_λ, then there exists φ_0, $\varphi_1 \in \mathrm{L}^1_{\mu_\lambda}(X,\mathbb{R}^+)$ such that:

$$\mu_0 = \varphi_0 \ \mu_\lambda, \quad \mu_1 = \varphi_1 \ \mu_\lambda.$$

Let $\beta \in B_r(t_0)$. Fix α_0, α_1 being ε-optimal strategies for

$$\inf_{\alpha\in A(t_0)} \int_X \int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)}) \, d\mu_i(x)dP_\beta(\omega) \quad i = 1,2.$$

Let τ_0, τ_1 being delays associated with α_0 and α_1. We build a new random strategy $(([0,1], \mathcal{B}([0,1]), L^1_{[0,1]}), \alpha_\lambda)$ with delay $\min(\tau_0,\tau_1)$ as follows:

$$\alpha_\lambda(x,\omega,v) := \begin{cases} \alpha_0(x,v) & \text{if } \omega \in [0,(1-\lambda)\varphi_0(x)] \\ \alpha_1(x,v) & \text{if } \omega \in](1-\lambda)\varphi_0(x), 1]. \end{cases}$$

Then:

$$\begin{aligned}
&\inf_{\alpha\in A_r(t_0)} \int_{\Omega_\alpha\times\Omega_\beta}\int_X g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot),\beta(\omega',\cdot)})\ dP_\alpha(\omega)\ dP_\beta(\omega')\ d\mu_\lambda(x)\\
\le\ & \int_{[0,1]\times\Omega_\beta}\int_X g(X_T^{t_0,\Phi(x),\alpha_\lambda(x,\omega,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ d\mu_\lambda(x)\\
=\ & \int_X\int_{[0,(1-\lambda)\varphi_0(x)]}\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_0(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ d\mu_\lambda(x)\\
&\quad +\int_X\int_{](1-\lambda)\varphi_0(x),1]}\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_1(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ d\mu_\lambda(x)\\
=\ & (1-\lambda)\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_0(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ \varphi_0(x)d\mu_\lambda(x)\\
&\quad +\lambda\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_1(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ \varphi_1(x)d\mu_\lambda(x)\\
=\ & (1-\lambda)\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_0(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ d\mu_0(x)\\
&\quad +\lambda\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha_1(x,\cdot),\beta(\omega',\cdot)})\ d\omega\ dP_\beta(\omega')\ d\mu_1(x)\\
\le\ & (1-\lambda)\inf_{\alpha\in A(t_0)}\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)})\ d\mu_0(x)dP_\beta(\omega)\\
&\quad +\lambda\inf_{\alpha\in A(t_0)}\int_X\int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)})\ d\mu_1(x)dP_\beta(\omega)+2\varepsilon.\\
\le\ & (1-\lambda)\mathcal{V}_r^-(t_0,\Phi,\mu_0)+\lambda\mathcal{V}_r^-(t_0,\Phi,\mu_1)+2\varepsilon.
\end{aligned}$$

Taking the supremum in β and sending ε to 0 gives the result.

QED

Both next Lemmas show some Lipschitz continuity properties of $\mathcal{V}_r^\pm$.

Lemma 4 *There exists $C>0$ such that for any $t,s\in[0,T]$, $\mu_0\in\Delta(X)$, $\Phi,\Psi\in \mathrm{L}^2_{\mu_0}(X,X)$,*

$$|\mathcal{V}_r^\pm(t,\Phi,\mu_0)-\mathcal{V}_r^\pm(s,\Psi,\mu_0)|\le C(|t-s|+\int_{\mathbb{R}^N}|\Phi(x)-\Psi(x)|d\mu_0(x)).$$

Proof: The Lipschitz regularity in t being standard we consider the case where $s=t=t_0$. We only make the proof for $\mathcal{V}_r^+$. By definition, we have:

$$\begin{aligned}
\mathcal{V}_r^+(t_0,\Phi,\mu_0)-\mathcal{V}_r^+(t_0,\Psi,\mu_0) &= \inf_{\alpha\in A_r(t_0)}\sup_{\beta\in B(t_0)}\int_{\Omega_\alpha}\int_{\mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ d\mu_0(x)dP_\alpha(\omega)\\
&\quad -\inf_{\alpha\in A_r(t_0)}\sup_{\beta\in B(t_0)}\int_{\Omega_\alpha}\int_{\mathbb{R}^N} g(X_T^{t_0,\Psi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ d\mu_0(x)dP_\alpha(\omega).
\end{aligned}$$

Let $\varepsilon > 0$ and $\alpha \in A_r(t_0)$ an ε-optimal strategy for $\mathcal{V}_r^+(t_0, \Psi, \mu_0)$ so that:

$$\mathcal{V}_r^+(t_0, \Phi, \mu_0) - \mathcal{V}_r^+(t_0, \Psi, \mu_0) \leq \sup_{\beta \in B(t_0)} \mathcal{J}(t_0, \Phi, \mu_0, \alpha, \beta) - \sup_{\beta \in B(t_0)} \mathcal{J}(t_0, \Psi, \mu_0, \alpha, \beta) + \varepsilon.$$

Then taking $\beta \in B(t_0)$ an ε-optimal strategy for $\sup_{\beta \in B(t_0)} \mathcal{J}(t_0, \Phi, \mu_0, \alpha, \beta)$ and using (3), we obtain:

$$\begin{aligned}\mathcal{V}_r^+(t_0, \Phi, \mu_0) - \mathcal{V}_r^+(t_0, \Psi, \mu_0) &\leq \mathcal{J}(t_0, \Phi, \mu_0, \alpha, \beta) - \mathcal{J}(t_0, \Psi, \mu_0, \alpha, \beta) + 2\varepsilon \\ &\leq CLip(g) \int_{\mathbb{R}^N} |\Phi(x) - \Psi(x)| \, d\mu_0(x) + 2\varepsilon.\end{aligned}$$

We denote $CLip(g)$ again by C.

QED

Remark 1 *The previous result implies in particular that the restriction of* $\mathcal{V}_r^+$ *to* $\mathcal{C}(X, X)$ *is Lipschitz for the norm* $\| \cdot \|_\infty$.

Lemma 5 *Fix* $p \in [1, +\infty)$. *There exists* $C > 0$ *such that, for any* $\Phi \in Lip(X, X)$, $\mu_0, \ \mu_1 \in \Delta(X)$:

$$|\mathcal{V}_r^\pm(t_0, \Phi, \mu_0) - \mathcal{V}_r^\pm(t_0, \Phi, \mu_1)| \leq Lip(\Phi) C W_p(\mu_0, \mu_1).$$

The proof is similar to [14], Proposition 3.
Proof: We will only prove the result for $\mathcal{V}_r^+$. We argue as in [14]. Fix $\varepsilon > 0$. Let $((\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha), \alpha)$ be a random strategy which is ε-optimal for $\mathcal{V}_r^+(t_0, \Phi, \mu_0)$: namely

$$(15) \qquad \int_{\Omega_\alpha} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0, \Phi(x), \alpha(x, \omega, \cdot)\beta(\cdot)}) \, d\mu_0(x) dP_\alpha(\omega) \leq \mathcal{V}_r^+(t_0, \Phi, \mu_0) + \varepsilon.$$

Fix $\beta \in B_r(t_0)$. Let γ be an optimal plan for $W_p(\mu_0, \mu_1)$. Then we disintegrate the measure γ with respect to μ_1 as follows

$$d\gamma(x, y) = d\gamma_y(x) d\mu_1(y).$$

There exists a map $\xi : \ (y, \omega') \in X \times [0,1]^N \mapsto \xi(y, \omega') \in \mathbb{R}^N$ such that

$$\xi(y, \cdot) \sharp \mathcal{L}^N = \gamma_y \text{ for } \mu_1\text{-almost all } y, \ \text{ and } W_2^2(\mathcal{L}^N, \gamma_y) = \int_{[0,1]^N} |\omega' - \xi(y, \omega')|^2 d\omega'.$$

It has been proven in [14] that this map ξ is measurable. This enables us to define the following random strategy for the first player

$$\tilde{\alpha} : (y, \omega, \omega', v) \in \mathbb{R}^N \times \Omega_\alpha \times [0,1]^N \times \mathcal{V}(t_0) \mapsto \alpha(\xi(y, \omega'), \omega, v) \in \mathcal{U}(t_0).$$

Then for any $\beta \in B(t_0)$ we have

$$\int_{\Omega_\alpha \times [0,1]^N} \int_{\mathbb{R}^N} g(X_T^{t_0, \Phi(y), \tilde{\alpha}(y, \omega, \omega', \cdot)\beta(\cdot)}) \, d\mu_1(y) dP_\alpha(\omega) d\omega'$$

$$= \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N} g(X_T^{t_0,\Phi(y),\alpha(x,\omega,\cdot)\beta(\cdot)})\ dP_\alpha(\omega) d\gamma_y(x) d\mu_1(y)$$

(Using Fubini Theorem and the definition of $\tilde{\alpha}$)

$$= \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N} g(X_T^{t_0,\Phi(y),\alpha(x,\omega,\cdot)\beta(\cdot)})\ dP_\alpha(\omega) d\gamma(x,y)$$

$$\leq \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ dP_\alpha(\omega) d\gamma(x,y) + C Lip(g) \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Phi(x) - \Phi(y)| d\gamma(x,y)$$

$$= \int_{\Omega_\alpha \times \mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ dP_\alpha(\omega) d\mu_0(x) + C Lip(g) \int_{\mathbb{R}^N \times \mathbb{R}^N} |\Phi(x) - \Phi(y)| d\gamma(x,y)$$

$$\leq \int_{\Omega_\alpha \times \mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\omega,\cdot)\beta(\cdot)})\ dP_\alpha(\omega) d\mu_0(x) + C Lip(\Phi) Lip(g) W_p(\mu_0, \mu_1,$$

(the last upper bound is due to Hölder inequality).
Hence by passing to the supremum over β, since ε is arbitrary, this yields

$$\mathcal{V}_r^+(t_0, \Phi, \mu_1) \leq \mathcal{V}_r^+(t_0, \Phi, \mu_0) + C Lip(g) Lip(\Phi) W_p(\mu_0, \mu_1).$$

Interchanging μ_0 and μ_1, the proof is complete.

QED

Remark 2 *In view of (12) taking in particular $\Phi = Id$ in the above result, we obtain that $\mu \mapsto V_r^\pm(t, \mu)$ is Lipschitz continuous with respect to the distance W_p uniformely in $t \in [0, T]$.*

Lemma 6 *For all $t_0 \in [0, T]$ and $\Phi \in \mathcal{C}(X, X)$, the maps $\mu_0 \mapsto \mathcal{V}_r^\pm(t_0, \Phi, \mu_0)$ are continuous with respect to W_2 (hence uniformly continuous since $\Delta(X)$ is compact).*

Proof: Again we only show the result for $\mathcal{V}_r^+$. Let $\Phi \in \mathcal{C}(X, X)$ and take $(\Phi_n)_n$ a sequence of Lipschitz functions converging uniformly to Φ. Let $\varepsilon > 0$. It exists $k \in \mathbb{N}$ such that $\|\Phi_k - \Phi\|_\infty \leq \frac{\varepsilon}{3C}$ where C is the constant of Lemma 4. So by Lemma 4:

$$\forall \mu \in \Delta(X), \quad |\mathcal{V}_r^+(t, \Phi, \mu) - \mathcal{V}_r^+(t, \Phi_k, \mu)| \leq \frac{\varepsilon}{3}.$$

Taking any μ_0, $\mu_1 \in \Delta(X)$ such that $W_p(\mu_0, \mu_1) \leq \frac{\varepsilon}{3} \frac{1}{C \times (Lip(\Phi_k) + 1)}$, we have by Lemma 5:

$$\begin{aligned} |\mathcal{V}_r^+(t, \Phi, \mu_0) - \mathcal{V}_r^+(t, \Phi, \mu_1)| &\leq |\mathcal{V}_r^+(t, \Phi, \mu_0) - \mathcal{V}_r^+(t, \Phi_k, \mu_0)| + |\mathcal{V}_r^+(t, \Phi_k, \mu_0) - \mathcal{V}_r^+(t, \Phi_k, \mu_1)| \\ &\quad + |\mathcal{V}_r^+(t, \Phi_k, \mu_1) - \mathcal{V}_r^+(t, \Phi, \mu_1)| \\ &\leq \frac{2\varepsilon}{3} + C(Lip(\Phi_k) + 1) W_2(\mu_0, \mu_1) \leq \varepsilon. \end{aligned}$$

The proof is complete.

QED

Remark 3 *In view of the previous result, the main result in [14] implies that* $\mathcal{V}_r^+ = \mathcal{V}_r^-$ *Also this enables us to obtain a result for pure strategies extended values* $\mathcal{V}^+$ *and* $\mathcal{V}^-$ *defined by*

$$\mathcal{V}^+(t_0, \Phi, \mu_0) = \inf_{\alpha \in A(t_0)} \sup_{\beta \in B(t_0)} \mathcal{J}(t_0, \Phi, \mu_0, \alpha, \beta),\ \mathcal{V}^-(t_0, \Phi, \mu_0) = \sup_{\beta \in B(t_0)} \inf_{\alpha \in A(t_0)} \mathcal{J}(t_0, \Phi, \mu_0, \alpha, \beta).$$

From [14] we have all $\mu_0 \in \Delta(X)$ *and* $\Phi \in \mathrm{L}^2_{\mu_0}(X, X)$ *such that* $\Phi\sharp\mu_0$ *has no atom,*

$$\mathcal{V}_r^{\pm}(t_0, \Phi, \mu_0) = \mathcal{V}^{\pm}(t_0, \Phi, \mu_0).$$

We end our section by showing the nonemptiness of the convex subdifferential of the value.

Lemma 7 *For all* $t \in [0, T]$ *and* $\Phi \in \mathcal{C}(X, X)$*, the maps* $\mu \mapsto \mathcal{V}_r^{\pm}(t, \Phi, \mu)$ *has a nonempty convex subdifferential at every point* $\bar{\mu} \in \Delta(X)$*:*

$$\emptyset \neq \partial_- \mathcal{V}_r^{\pm}(t, \Phi, \bar{\mu}) := \{\phi \in C(X),\ \forall \mu \in \Delta(X),\ \mathcal{V}_r^{\pm}(t, \Phi, \mu) - \mathcal{V}_r^{\pm}(t, \Phi, \bar{\mu}) \geq \int_X \phi d(\mu - \bar{\mu})\}.$$

Proof: Again we only show the result for $\mathcal{V}_r^+$. Fix $\bar{\mu} \in \Delta(X)$, $\Phi \in C(X, X)$ and $t \in [0, T]$. From remark 2, we know that $V_r^+(t, \cdot) : \Delta(X) \mapsto \mathbb{R}$ are convex and Lipschitz continuous with respect to the Wasserstein distance W_1.

We claim that $\partial_- V_r^+(t, \Phi\sharp\bar{\mu}) \neq \emptyset$. Set $\mu_0 := \Phi\sharp\bar{\mu}$ and let us define

$$Z := \{\nu - \mu_0,\ \ \nu \in \Delta(X)\,\},$$

which is a convex compact subset of the vectorial space $M_0(X)$. Here $M_0(X)$ is the set of signed measures of total mass zero on X equipped with the Monge-Kantorovich norm :

$$\forall \sigma \in M_0(X),\ \|\sigma\|_{MK} := \sup\{\int_X \phi(x) d(\sigma^+ - \sigma^-)(x),\ \phi \in C(X),\ Lip(\phi) \leq 1\,\}.$$

Recall that for any $(m, m') \in \Delta(X)^2$ we have $\|m - m'\|_{MK} = W_1(m, m')$ (cf [30]). We define now the map $G : \nu - \mu_0 \in Z \mapsto V_r^+(t, \nu)$ which is bounded convex and C-Lipschitz for the Monge-Kantorovich norm in view of Remark 2.

The function G being C-Lipschitz and convex on the convex compact set Z, we may define the following extension to $M_0(X)$

$$\bar{G}(\sigma) := \inf_{\rho \in C}\{\,G(\rho) + C\|\rho - \sigma\|_{MK}\,\},\ \forall \sigma \in M_0(X),$$

which is still convex and Lipschitz.

Recalling that $M_0(X)$ is in duality with $Lip(X)/\mathbb{R}$ (the space of Lipschitz function up to a constant) (cf [21] or [7]), we deduce that the convex Lipschitz function G which is bounded in a neighborhhod of Z has a nonempty subdifferential at $0 \in Z$ (cf [16]). So there exists $\xi \in Lip(X)$ such that in particular

$$\forall \rho \in C,\ \bar{G}(\rho) = G(\rho) \geq G(0) + < \rho, \xi > .$$

Taking $\rho = \Phi\sharp\mu - \mu_0$ this yields

$$\forall \mu \in \Delta(X), V_r^+(t, \Phi\sharp\mu) - V_r^+(t, \Phi\sharp\bar{\mu}) \geq \int_X \xi d(\Phi\sharp\mu - \Phi\sharp\bar{\mu}),$$

which proves our claim.

Using Lemma 2, we get

$$\forall \mu \in \Delta(X), \mathcal{V}_r^+(t, \Phi, \mu) - \mathcal{V}_r^+(t, \Phi, \bar{\mu}) \geq \int_X \xi \circ \Phi d(\mu - \bar{\mu}).$$

Hence $\xi \circ \Phi \in \partial_- \mathcal{V}_r^+(t, \Phi, \bar{\mu})$ which completes the proof.

QED

3 Subdynamic Programming Principles

3.1 Subdynamic principle for $\mathcal{V}_r^+$

Proposition 2 *Let $\mu_0 \in \Delta(X)$ and $\Phi \in L^2_{\mu_0}(X, X)$, $t_1 \in]t_0, T]$, it holds:*

$$\mathcal{V}_r^+(t_0, \Phi, \mu_0) \leq \inf_{\alpha \in A_c(t_0)} \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1, X_{t_1}^{t_0, \cdot, \alpha, v} \circ \Phi, \mu_0)$$

with $A_c(t_0) := \{\alpha \in A(t_0) :\ \alpha$ is constant in the space variable $x\}$.

Remark 4 *As the map $x \mapsto X_{t_1}^{t_0, \cdot, \alpha, v}$ is Lipschitz and because X is invariant, $X_{t_1}^{t_0, \cdot, \alpha, v} \circ \Phi$ belongs to $L^2_{\mu_0}(X, X)$.*

Proof: Let $\varepsilon > 0$ and $\alpha_0 \in A_c(t_0)$ such that:

$$\text{(16)} \qquad \inf_{\alpha \in A_c(t_0)} \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1, X_{t_1}^{t_0, \cdot, \alpha, v} \circ \Phi, \mu_0) \geq \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1, X_{t_1}^{t_0, \cdot, \alpha_0, v} \circ \Phi, \mu_0) - \varepsilon.$$

Let also $((\Omega_1, \mathcal{F}_1, P_1), \alpha_1)$ be any element of $A_r(t_1)$.
We introduce a new strategy $((\Omega_1, \mathcal{F}_1, P_1), \bar{\alpha}) \in A_r(t_0)$ built by gluing α_0 and α_1 in the following way:
(17)

$$\forall (x, \omega, v, s) \in X \times \Omega_1 \times \mathcal{V}(t_0) \times [t_0, T], \quad \bar{\alpha}(x, \omega, v)(s) = \begin{cases} \alpha_0(v)(s) & \text{if } s \in [t_0, t_1[, \\ \alpha_1(x, \omega, v\lfloor_{[t_1, T]})(s) & \text{else.} \end{cases}$$

This new strategy satisfies

$$\mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, v) = \mathcal{J}(t_1, X_{t_1}^{t_0, \cdot, \alpha_0, v} \circ \Phi, \mu_0, \alpha_1, v\lfloor_{[t_1, T]}) \quad \forall v \in \mathcal{V}(t_0).$$

We will now use the following lemma:

Lemma 8 *Let $(\alpha_0, \alpha_1, \beta) \in A_c(t_0) \times A_r(t_1) \times B(t_0)$ and $\bar{\alpha}$ defined by (17), then it exists $\tilde{\beta} \in B(t_1)$ such that:*

$$\mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, \beta) = \mathcal{J}(t_1, X_{t_1}^{t_0, \cdot, \alpha_0, \beta} \circ \Phi, \mu_0, \alpha_1, \tilde{\beta}).$$

Then, take $\bar{\beta} \in B(t_0)$ such that

$$\sup_{\beta \in B(t_0)} \mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, \beta) \leq \mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, \bar{\beta}) + \varepsilon,$$

so that:

$$\mathcal{V}_r^+(t_0, \Phi, \mu_0) \leq \mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, \bar{\beta}) + \varepsilon. \tag{18}$$

Then using Lemma 8, there exists $\tilde{\beta} \in B(t_1)$ such that:

$$\mathcal{V}_r^+(t_0, \Phi, \mu_0) \leq \mathcal{J}(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,\bar{\beta}} \circ \Phi,\ \mu_0, \alpha_1, \tilde{\beta}) + \varepsilon \leq \sup_{\beta_1 \in B(t_1)} \mathcal{J}(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,\bar{\beta}} \circ \Phi,\ \mu_0, \alpha_1, \beta_1) + \varepsilon,$$

this inequality holds true for any $\alpha_1 \in A_r(t_1)$, so we get:

$$\begin{aligned} \mathcal{V}_r^+(t_0, \Phi, \mu_0) &\leq \inf_{\alpha \in A_r(t_1)} \sup_{\beta_1 \in B(t_1)} \mathcal{J}(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,\bar{\beta}} \circ \Phi,\ \mu_0, \alpha, \beta_1) + \varepsilon, \\ &= \mathcal{V}_r^+(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,\bar{\beta}} \circ \Phi,\ \mu_0) + \varepsilon \leq \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,v} \circ \Phi,\ \mu_0) + \varepsilon. \end{aligned}$$

We conclude using (16):

$$\begin{aligned} \mathcal{V}_r^+(t_0, \Phi, \mu_0) &\leq \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,v} \circ \Phi,\ \mu_0) + \varepsilon \\ &\leq \inf_{\alpha \in A_c(t_0)} \sup_{v \in \mathcal{V}(t_0)} \mathcal{V}_r^+(t_1,\ X_{t_1}^{t_0,\cdot,\alpha,v} \circ \Phi,\ \mu_0) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the proof is complete.

QED

Proof of Lemma 8. To $(\alpha_0, \beta) \in A(t_0) \times B(t_0)$ we associate $(\bar{u}, \bar{v}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that:

$$\alpha_0(\bar{v}) = \bar{u}, \quad \beta(\bar{u}) = \bar{v}.$$

We build a new strategy $\tilde{\beta} \in B(t_1)$ by setting [3]:

$$\tilde{\beta}(u) := \beta(\bar{u} \oplus u) \quad \forall u \in \mathcal{U}(t_1),$$

where $\bar{u} \oplus u$ is defined by

$$\bar{u} \oplus u(s) = \bar{u}(s) \text{ if } s \in [t_0, t_1] \text{ and } \bar{u} \oplus u(s) = u(s) \text{ else.}$$

To $(\alpha_1, \tilde{\beta})$ we associate the measurable map $(x, \omega) \in X \times \Omega_1 \mapsto (u_1^{x,\omega}, v_1^{x,\omega}) \in \mathcal{U}(t_1) \times \mathcal{V}(t_1)$ defined as:

$$\alpha_1(x, \omega, v_1^{x,\omega}) = u_1^{x,\omega}, \quad \tilde{\beta}(u_1^{x,\omega}) = v_1^{x,\omega},$$

[3] As a consequence of the definition $\tilde{\beta}(u)(s) = \beta(\bar{u})(s) = \bar{v}(s)\ \forall s \in [t_0, t_1 + \tau]$ for some $\tau > 0$.

(observe that we have $v_1^{x,\omega} = \bar{v}$ on $[t_1, t_1+\tau]$). Now we notice that the controls $\bar{u} \oplus u_1^{x,\omega}$ and $\bar{v} \oplus v_1^{x,\omega}$ satisfy:

$$\beta(\bar{u} \oplus u_1^{x,\omega}) = \bar{v} \oplus v_1^{x,\omega}, \quad \bar{\alpha}(x,\omega,\bar{v} \oplus v_1^{x,\omega}) = \bar{u} \oplus u_1^{x,\omega}. \tag{19}$$

Indeed by the very definition of $\tilde{\beta}$ and $\bar{\alpha}$:

$$\beta(\bar{u} \oplus u_1^{x,\omega})(s) = \begin{cases} \beta(\bar{u})(s) = \bar{v}(s) & \forall s \in [t_0, t_1+\tau] \\ \tilde{\beta}(u_1^{x,\omega})(s) = v_1^{x,\omega}(s) & \text{in } [t_1, T]; \end{cases}$$

$$\bar{\alpha}(x,\omega,\bar{v} \oplus v_1^{x,\omega})(s) = \begin{cases} \alpha_0(\bar{v})(s) = \bar{u}(s) & \text{if } s \in [t_0, t_1], \\ \alpha_1(x,\omega,v_1^{x,\omega})(s) = u_1^{x,\omega}(s) & \text{elsewhere.} \end{cases}$$

Let us now compute $\mathcal{J}(t_0, \Phi, \mu_0, \bar{\alpha}, \beta)$:

$$\mathcal{J}(t_0,\Phi,\mu_0,\bar{\alpha},\beta) := \int_{\Omega_1}\int_X g(X_T^{t_0,\Phi(x),\bar{\alpha}(x,\omega,\cdot),\beta})\, dP_1(\omega)d\mu_0(x) =$$

$$\int_{\Omega_1}\int_X g(X_T^{t_0,\Phi(x),\bar{u}\oplus u_1^{x,\omega},\bar{v}\oplus v_1^{x,\omega}})\, dP_1(\omega)d\mu_0(x) = \int_{\Omega_1}\int_X g\left(X_T^{t_1,\ X_{t_1}^{t_0,\Phi(x),\bar{u},\bar{v}},\ u_1^{x,\omega},v_1^{x,\omega}}\right)\, dP_1(\omega)d\mu_0(x)$$

$$= \int_{\Omega_1}\int_X g\left(X_T^{t_1,\ X_{t_1}^{t_0,\Phi(x),\alpha_0,\beta},\ \alpha_1(x,\omega,\cdot),\ \tilde{\beta}}\right)\, dP_1(\omega)d\mu_0(x) = \mathcal{J}(t_1,\ X_{t_1}^{t_0,\cdot,\alpha_0,\beta}\circ\Phi,\ \mu_0,\alpha_1,\tilde{\beta}).$$

QED

3.2 Dual subdynamic principle for $\mathcal{V}_r^-$

Following [10, 17], we introduce the Fenchel conjugate of $\mathcal{V}_r^-$:

$$(\mathcal{V}_r^-)^*(t_0,\Phi,\varphi) := \sup_{\mu_0\in\Delta(X)}\left\{\int_X \varphi d\mu_0 - \mathcal{V}_r^-(t_0,\Phi,\mu_0)\right\},\ \forall (t_0,\Phi,\varphi)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{C}(X).$$

Lemma 9

$$(\mathcal{V}_r^-)^*(t_0,\Phi,\varphi) = \inf_{\beta\in B_r(t_0)}\sup_{\alpha\in A_c(t_0)}\sup_{x\in X}\left\{\varphi(x) - \int_{\Omega_\beta} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)}\right)\, dP_\beta(\omega)\right\}. \tag{20}$$

Let us denote by $z = z(t_0,\Phi,\varphi)$ the right-hand side of (20). First show the following

Lemma 10 *For any fixed $\Phi \in \mathcal{C}(X,X)$, $t_0 \in [0,T]$, $z : \varphi \in \mathcal{C}(X) \mapsto z(t_0,\Phi,\varphi)$ is convex and lower semi-continuous for the uniform topology.*

Proof: Let φ_0, φ_1 in $\mathcal{C}(X)$ and $\lambda \in [0,1]$, we set $\varphi_\lambda := (1-\lambda)\varphi_0 + \lambda\varphi_1$.

Let $((\Omega_i, \mathcal{F}_i, P_i), \beta_i)$ be ε-optimal strategies for $z(t_0,\Phi,\varphi_i)$, $i = 0, 1$. We set:

$$\Omega_\lambda = \Omega_0\times\Omega_1\times[0,1], \quad \mathcal{F}_\lambda = \mathcal{F}_0\times\mathcal{F}_1\times\mathcal{B}([0,1]), \quad P_\lambda = P_0\times P_1\times\mathcal{L}^1_{[0,1]},$$

$$\forall(\omega_0,\omega_1,\omega)\in\Omega_0\times\Omega_1\times[0,1], \quad \beta(\omega_0,\omega_1,\omega,\cdot) = \begin{cases} \beta_0(\omega_0,\cdot) & \text{if } \omega\in[0,1-\lambda] \\ \beta_1(\omega_1,\cdot) & \text{if } \omega\in]1-\lambda,1]. \end{cases}$$

It holds:

$$z(t_0,\Phi,\varphi_\lambda) \le \sup_{\alpha\in A_c(t_0)} \max_{x\in X} \left\{ \varphi_\lambda(x) - \int_{\Omega_\lambda} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega_0,\omega_1,\omega,\cdot)} \right) \, dP_\lambda(\omega_0,\omega_1,\omega) \right\}$$

$$= \sup_{\alpha\in A_c(t_0),\ x\in X} \left\{ \varphi_\lambda(x) - \int_{\Omega_0\times\Omega_1} \int_0^1 g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega_0,\omega_1,\omega,\cdot)} \right) \, dP_0(\omega_0) dP_1(\omega_1) d\omega \right\}$$

$$= \sup_{\alpha\in A_c(t_0),\ x\in X} \left\{ \varphi_\lambda(x) - (1-\lambda) \int_{\Omega_0} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta_0(\omega_0,\cdot)} \right) \, dP_0(\omega_0) \right.$$

$$\left. -\lambda \int_{\Omega_0} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta_1(\omega_1,\cdot)} \right) \, dP_1(\omega_1) \right\}$$

$$\le (1-\lambda) \sup_{\alpha\in A_c(t_0),\ x\in X} \left\{ \varphi_0(x) - (1-\lambda) \int_{\Omega_0} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta_0(\omega_0,\cdot)} \right) \, dP_0(\omega_0) \right\}$$

$$+\lambda \sup_{\alpha\in A_c(t_0),\ x\in X} \left\{ \varphi_1(x) - \int_{\Omega_0} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta_1(\omega_1,\cdot)} \right) \, dP_1(\omega_1) \right\}$$

$$\le (1-\lambda) z(t_0,\Phi,\varphi_0) + \lambda z(t_0,\Phi,\varphi_1) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ gives the wished convexity result.

Now we show z is l.s.c. Assume $\|\varphi_n - \varphi\|_\infty \to 0$. We have clearly

$$\begin{aligned} z(t_0,\Phi,\varphi_n) &= \inf_{\beta\in B_r(t_0)} \sup_{\alpha\in A_c(t_0)} \max_{x\in X} \left\{ \varphi_n(x) - \int_{\Omega_\beta} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)} \right) \, dP_\beta(\omega) \right\} \\ &\ge z(t_0,\Phi,\varphi) - \|\varphi_n - \varphi\|_\infty. \end{aligned}$$

Taking the liminf as n tends to $+\infty$ gives the result.

QED

Proof of Lemma 9: Step 1: Let us show $(\mathcal{V}_r^-)^* \le z$. Indeed:

$$(\mathcal{V}_r^-)^*(t_0,\Phi,\varphi) := \sup_{\mu_0\in\Delta(X)} \left\{ \int_X \varphi d\mu_0 - \mathcal{V}_r^-(t_0,\Phi,\mu_0) \right\}$$

$$= \sup_{\mu_0\in\Delta(X)} \left\{ \int_X \varphi d\mu_0 - \sup_{\beta\in B_r(t_0)} \inf_{\alpha\in A(t_0)} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0,\Phi(x),\alpha(x,\cdot)\beta(\omega,\cdot)}) \, d\mu_0(x) dP_\beta(\omega) \right\}$$

$$= \sup_{\mu_0\in\Delta(X)} \inf_{\beta\in B_r(t_0)} \sup_{\alpha\in A(t_0)} \left\{ \int_X \left[\varphi(x) - \int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha(x,\cdot)\beta(\omega,\cdot)}) \, dP_\beta(\omega) \right] d\mu_0(x) \right\}$$

$$\le \sup_{\mu_0\in\Delta(X)} \inf_{\beta\in B_r(t_0)} \left\{ \int_X \sup_{\alpha\in A_c(t_0)} \sup_{y\in X} \left[\varphi(y) - \int_{\Omega_\beta} g(X_T^{t_0,\Phi(y),\alpha(\cdot)\beta(\omega,\cdot)}) \, dP_\beta(\omega) \right] d\mu_0(x) \right\}$$

$$= \inf_{\beta\in B_r(t_0)} \sup_{\alpha\in A_c(t_0), x\in X} \varphi(x) - \int_{\Omega_\beta} g(X_T^{t_0,\Phi(x),\alpha(x,\cdot)\beta(\omega,\cdot)}) \, dP_\beta(\omega) = z(t_0,\Phi,\varphi).$$

Step 2: Now, we show $(\mathcal{V}_r^-)^* \ge z^{**}$.

For any $\alpha \in A_c(t_0)$ and $\beta \in B_r(t_0)$, we introduce the function $\xi_{\alpha,\beta} \in \mathcal{C}(X)$ defined by :

$$\xi_{\alpha,\beta}(x) = \int_{\Omega_\beta} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)} \right) \, dP_\beta(\omega).$$

We have for any $\alpha \in A_c(t_0)$ and any $\beta \in B_r(t_0)$ (cf (3):

$$\xi_{\alpha,\beta}(x) - \xi_{\alpha,\beta}(x') \leq CLip(g)|\Phi(x) - \Phi(x')|.$$

Hence $\xi_{\alpha,\beta}$ is uniformly continuous with the same modulus of continuity that Φ. Moreover, by taking the infimum in α, and α_0 ε-optimal for $\inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x')$:

$$\left(\inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x) - \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x')\right) \;\leq\; \xi_{\alpha_0,\beta}(x) - \xi_{\alpha_0,\beta}(x') + \varepsilon \;\leq\; CLip(g)|\Phi(x) - \Phi(x')| + \varepsilon,$$

this shows that $\inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}$ is also continuous. We have:

$$z(t_0, \Phi, \varphi) = \inf_{\beta\in B_r(t_0)} \max_{x\in X} \left\{\varphi(x) - \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x)\right\}.$$

So that z^* rewrites as:

$$\begin{aligned} z^*(t_0, \Phi, \mu_0) &= \sup_{\varphi\in\mathcal{C}(X), \beta\in B_r(t_0)} \inf_{x\in X} \left\{\langle\mu_0, \varphi\rangle - \varphi(x) + \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x)\right\} \\ &= \sup_{\beta\in B_r(t_0)} \sup_{\varphi\in\mathcal{C}(X)} \inf_{x\in X} \left\{\langle\mu_0, \varphi\rangle - \varphi(x) + \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}(x)\right\} \end{aligned}$$

Take $\varphi = \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}$, then we get:

$$\begin{aligned} z^*(t_0, \Phi, \mu_0) &\geq \sup_{\beta\in B_r(t_0)} \inf_{x\in X} \left\{\langle\mu_0, \inf_{\alpha\in A_c(t_0)} \xi_{\alpha,\beta}\rangle\right\} \\ &= \sup_{\beta\in B_r(t_0)} \inf_{x\in X} \left\{\int_X \inf_{\alpha\in A_c(t_0)} [\int_{\Omega_\beta} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)}\right) \, dP_\beta(\omega)] \, d\mu_0(x)\right\} \end{aligned}$$

Now, if $\mu_0 = \sum_{i=1}^n c_i\delta_{x_i}$ is any element of $\Delta(X)$ with finite support, we have:

$$\begin{aligned} \mathcal{V}_r^-(t_0, \Phi, \mu_0) &= \sup_{\beta\in B_r(t_0)} \inf_{\alpha_i\in A_c(t_0)} \sum_{i=1}^n c_i \int_{\Omega_\beta} g(X_T^{t_0,\Phi(x_i),\alpha_i(\cdot)\beta(\omega,\cdot)}) \, dP_\beta(\omega) \\ &= \sup_{\beta\in B_r(t_0)} \sum_{i=1}^n c_i \inf_{\alpha\in A_c(t_0)} \int_{\Omega_\beta} g(X_T^{t_0,\Phi(x_i),\alpha(\cdot)\beta(\omega,\cdot)}) \, dP_\beta(\omega) \\ &= \sup_{\beta\in B_r(t_0)} \inf_{x\in X} \left\{\int_X \inf_{\alpha\in A_c(t_0)} [\int_{\Omega_\beta} g\left(X_T^{t_0,\Phi(x),\alpha(\cdot),\beta(\omega,\cdot)}\right) \, dP_\beta(\omega)] \, d\mu_0(x)\right\}. \end{aligned}$$

So that for any μ_0 with finite support:

$$\mathcal{V}_r^-(t_0, \Phi, \mu_0) \leq z^*(t_0, \Phi, \mu_0).$$

Then by density, as $\mathcal{V}_r^-(t_0, \Phi, \cdot)$ is continuous for the weak star topology:

$$(\mathcal{V}_r^-)^*(t_0, \Phi, \varphi) = \sup_{\mu_0 \text{ with finite support}} \left\{\int_\Omega \varphi d\mu_0 - \mathcal{V}_r^-(t_0, \Phi, \mu_0)\right\} \geq z^{**}(t_0, \Phi, \varphi).$$

This together with step 1 and Lemma 10 imply $z^{**} = z = (\mathcal{V}_r^-)^*$. The proof is complete.

QED

Proposition 3 *For any* $(t_0, \Phi, \varphi) \in [0,T] \times \mathcal{C}(X,X) \times \mathcal{C}(X)$*, it holds:*

$$(\mathcal{V}_r^-)^*(t_0, \Phi, \varphi) \leq \inf_{\beta \in B(t_0)} \sup_{u \in \mathcal{U}(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0, \Phi(\cdot), u, \beta}, \varphi).$$

Proof: Set for all $(t_0, \Phi, \varphi, \alpha, \beta) \in [0,T] \times \mathcal{C}(X,X) \times \mathcal{C}(X) \times A_c(t_0) \times B_r(t_0)$:

$$\mathcal{G}(t_0, \Phi, \varphi, \alpha, \beta) = \sup_{x \in X} \left\{ \varphi(x) - \int_{\Omega_\beta} g(X_T^{t_0, \Phi(x), \alpha, \beta(\omega, \cdot)}) \; dP_\beta(\omega) \right\}.$$

We mimic the proof of Proposition 2. Let $\varepsilon > 0$ and $\beta_0 \in B(t_0)$ such that:

$$\inf_{\beta \in B(t_0)} \sup_{\alpha \in A_c(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0, \Phi(\cdot), \alpha, \beta}, \varphi) \geq \sup_{\alpha \in A_c(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0, \Phi(\cdot), \alpha, \beta_0}, \varphi) - \varepsilon. \tag{21}$$

Let also $((\Omega_1, \mathcal{F}_1, P_1), \beta_1)$ be an element of $B_r(t_1)$. We introduce a new strategy $((\Omega_1, \mathcal{F}_1, P_1), \bar{\beta}) \in B_r(t_0)$ built by gluing β_0 and β_1 in the following way:

$$\forall (\omega, u, s) \in \Omega_1 \times \mathcal{U}(t_0) \times [t_0, T], \quad \bar{\beta}(\omega, u)(s) = \begin{cases} \beta_0(u)(s) & \text{if } s \in [t_0, t_1[, \\ \beta_1(\omega, u\lfloor_{[t_1,T]})(s) & \text{else.} \end{cases} \tag{22}$$

This new strategy satisfies

$$\mathcal{G}(t_0, \Phi, \varphi, u, \bar{\beta}) = \mathcal{G}(t_1, X_{t_1}^{t_0, \cdot, u, \beta_0} \circ \Phi, \varphi, u\lfloor_{[t_1,T]}, \beta_1) \quad \forall u \in \mathcal{U}(t_0).$$

We introduce the following lemma that can be proved similarly to Lemma 8:

Lemma 11 *Let* $(\beta_0, \beta_1, \alpha) \in B(t_0) \times B_r(t_1) \times A_c(t_0)$ *and* $\bar{\beta}$ *defined by (22), then it exists* $\tilde{\alpha} \in A_c(t_1)$ *such that:*

$$\mathcal{G}(t_0, \Phi, \varphi, \alpha, \bar{\beta}) = \mathcal{G}(t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta_0} \circ \Phi, \mu_0, \tilde{\alpha}, \beta_1).$$

Then, take $\bar{\alpha} \in A(t_0)$ such that

$$\sup_{\alpha \in A_c(t_0)} \mathcal{G}(t_0, \Phi, \varphi, \alpha, \bar{\beta}) \leq \mathcal{G}(t_0, \Phi, \varphi, \bar{\alpha}, \bar{\beta}) + \varepsilon,$$

so that, by (20):

$$(\mathcal{V}_r^-)^*(t_0, \Phi, \varphi) \leq \mathcal{G}(t_0, \Phi, \varphi, \bar{\alpha}, \bar{\beta}) + \varepsilon. \tag{23}$$

Then using Lemma 11, there exists $\tilde{\alpha} \in A_c(t_1)$ such that:

$$(\mathcal{V}_r^-)^*(t_0, \Phi, \varphi) \leq \mathcal{G}(t_1, X_{t_1}^{t_0, \cdot, \bar{\alpha}, \beta_0} \circ \Phi, \mu_0, \tilde{\alpha}, \beta_1) + \varepsilon \leq \sup_{\alpha_1 \in A_c(t_1)} \mathcal{G}(t_1, X_{t_1}^{t_0, \cdot, \bar{\alpha}, \beta_0} \circ \Phi, \mu_0, \alpha_1, \beta_1) + \varepsilon,$$

this inequality holds true for any $\beta_1 \in B_r(t_1)$, so we get again by (20):

$$\begin{aligned} (\mathcal{V}_r^-)^*(t_0, \Phi, \varphi) &\leq \inf_{\beta_1 \in B_r(t_1)} \sup_{\alpha_1 \in A_c(t_1)} \mathcal{G}(t_1, X_{t_1}^{t_0, \cdot, \bar{\alpha}, \beta_0} \circ \Phi, \mu_0, \alpha_1, \beta_1) + \varepsilon, \\ &= (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0, \Phi(\cdot), \bar{\alpha}, \beta_0}, \varphi) + \varepsilon \leq \sup_{u \in \mathcal{U}(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0, \Phi(\cdot), u, \beta_0}, \varphi) + \varepsilon. \end{aligned}$$

We conclude using (21):

$$
\begin{aligned}
(\mathcal{V}_r^-)^*(t_0,\Phi,\varphi) &\le \sup_{u\in\mathcal{U}(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0,\Phi(\cdot),u,\beta_0},\varphi)+\varepsilon \\
&\le \inf_{\beta\in B(t_0)} \sup_{\alpha\in A_c(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0,\Phi(\cdot),\alpha,\beta},\varphi)+2\varepsilon.
\end{aligned}
$$

QED

4 Hamilton Jacobi Isaacs equations

We introduce the following Hamiltonian defined for any $(\mu_0,\Phi_0,p_\Phi)\in\Delta(X)\times\mathcal{C}(X,X)\times\mathcal{C}(X,X)$ by:

$$
\mathcal{H}(\mu_0,\Phi_0,p_\Phi):=\inf_{u\in U}\sup_{v\in V}\int_X f(\Phi_0(x),u,v)\cdot p_\Phi(x)\,d\mu_0(x)=\sup_{v\in V}\inf_{u\in U}\int_X f(\Phi_0(x),u,v)\cdot p_\Phi(x)\,d\mu_0(x)
$$

and the Hamilton Jacobi Isaacs equation:

$$
\partial_t W+\mathcal{H}(\mu_0,\Phi_0,DW)=0 \tag{24}
$$

We will also need $\widehat{\mathcal{H}}$:

$$
\widehat{\mathcal{H}}(\mu_0,\Phi_0,p_\Phi):=-\mathcal{H}(\mu_0,\Phi_0,-p_\Phi)
$$

4.1 Subsolution and Dual supersolution

Hereafter, we give the appropriate definitions of superdifferential and solutions:

Definition 5 *Consider the function $w:[0,T]\times\mathcal{C}(X,X)\times\Delta(X)\mapsto\mathbb{R}$ and $w^*:[0,T]\times\mathcal{C}(X,X)\times\mathcal{C}(X)\mapsto\mathbb{R}$ be its Fenchel conjugate in the μ variable.*

- *Let $\delta>0$ and $(t_0,\Phi_0,\mu_0)\in]0,T[\times\mathcal{C}(X,X)\times\Delta(X)$. We say that $(p_t,p_\Phi)\in\mathbb{R}\times\mathcal{C}(X,X)$ belongs to the δ-superdifferential $D^+_\delta w(t_0,\Phi_0,\mu_0)$ to w at (t_0,Φ_0,μ_0) iff*

$$
\limsup_{\|\xi\|_\infty\to 0,\ t\to t_0}\frac{w(t,\Phi_0+\xi,\mu_0)-w(t_0,\Phi_0,\mu_0)-p_t(t-t_0)-\int_X\xi(x)\cdot p_\Phi(x)d\mu_0(x)}{\|\xi\|_\infty+|t-t_0|}\le\delta
$$

  (here $\|\xi\|_\infty$ stands for the norm of the uniform convergence of $\xi\in\mathcal{C}(X,X)$).

- *Let $\delta>0$, $\mu_0\in\Delta(X)$ and $(t_0,\Phi_0,\varphi)\in]0,T[\times\mathcal{C}(X,X)\times\mathcal{C}(X)$. We say that $(p_t,p_\Phi)\in\mathbb{R}\times\mathcal{C}(X,X)$ belongs to the δ-superdifferential $D^+_{\delta,\mu_0}w^*(t_0,\Phi_0,\varphi)$ to w^* at (t_0,Φ_0,φ) with respect to μ_0 iff*

$$
\limsup_{\|\xi\|_\infty\to 0,\ t\to t_0}\frac{w^*(t,\Phi_0+\xi,\varphi)-w^*(t_0,\Phi_0,\varphi)-p_t(t-t_0)-\int_X\xi(x)\cdot p_\Phi(x)d\mu_0(x)}{\|\xi\|_\infty+|t-t_0|}\le\delta.
$$

Definition 6 *For any $\mu_0 \in \Delta(X)$, the map $(t, \Phi, \mu) \in [0, T] \times \mathcal{C}(X, X) \times \Delta(X) \mapsto w(t, \Phi, \mu)$ is a viscosity subsolution to (24) iff it exists $C > 0$ such that, for all $\delta > 0$, all $(t_0, \Phi_0) \in]0, T[\times \mathcal{C}(X, X)$ and all $(p_t, p_\Phi) \in D^+_\delta w(t_0, \Phi_0, \mu_0)$, we have:*

$$p_t + \mathcal{H}(\mu_0, \Phi_0, p_\Phi) \geq -C\delta.$$

Definition 7 *For any $\mu_0 \in \Delta(X)$, the function $(t, \Phi, \mu) \in [0, T] \times \mathcal{C}(X, X) \times \Delta(X) \mapsto w(t, \Phi, \mu)$ is a viscosity dual supersolution to (24) iff it exists $C > 0$ such that, for all $\delta > 0$ and all $(t_0, \Phi_0, \mu_0, \varphi) \in]0, T[\times C(X, X) \times \Delta(X) \times C(X)$ and $(p_t, p_\Phi) \in D^+_{\delta,\mu_0} w^*(t_0, \Phi_0, \varphi)$:*

$$p_t + \widehat{\mathcal{H}}(\mu_0, \Phi_0, p_\Phi) \geq -C\delta.$$

We state now a comparison principle for the Hamilton Jacobi Isaacs equation:

Theorem 1 *For $i = 1, 2$, let $w_i : [0, T] \times \mathcal{C}(X, X) \times \Delta(X) \to \mathbb{R}$ be uniformly continuous bounded maps. We assume that:*

(H1) for any fixed $\mu \in \Delta(X)$, $w_i(\cdot, \cdot, \mu)$ is k-Lipschitz continuous with $k > 0$ i.e.:

$$|w_i(t, \Phi, \mu) - w_i(s, \Psi, \mu)| \leq k \left(|s - t| + \|\Phi - \Psi\|_{L^2_\mu} \right);$$

(H2) the map w_i is convex in the μ variable and $\partial_- w_i(t, \Phi, \mu) \neq \emptyset$ for all $(t, \Phi, \mu) \in [0, T] \times \mathcal{C}(X, X) \times \Delta(X)$.

(H3) for any $\mu_0 \in \Delta(X)$, w_1 is a subsolution of (24) and w_2 is a dual supersolution of (24);

(H4) the following equality holds: $w_1(T, \cdot, \cdot) = w_2(T, \cdot, \cdot)$.

Then for all $(t, \Phi, \mu) \in [0, T] \times C(X, X) \times \Delta(X)$:

$$w_1(t, \Phi, \mu) \leq w_2(t, \Phi, \mu).$$

Proof: Assume by contradiction that for some $\alpha > 0$ and some $(t_0, \Phi_0, \mu_0) \in [0, T[\times \mathcal{C}(X, X) \times \Delta(X)$ such that:

$$(w_2 - w_1)(t_0, \Phi_0, \mu_0) \leq -\frac{\alpha}{2}. \tag{25}$$

Denote by $C > 0$ the constant for which both inequalities of sub and super solutions holds. We choose $\eta > 0$ small enough such that

$$2T\eta < \frac{\alpha}{4} \text{ and } \eta k < \frac{\alpha}{4} \tag{26}$$

and then we choose $\varepsilon \in (0, 1)$ small enough for having

$$2(k + C)\varepsilon < \eta, \; k\varepsilon < T \text{ and } k(k + 1)\varepsilon < \frac{\alpha}{4}. \tag{27}$$

We consider the following map on $[0,T]^2 \times \mathcal{C}(X,X)^2 \times \Delta(X)$:

$$\theta(t,s,\Phi,\Psi,\mu) = w_2(s,\Psi,\mu) - w_1(t,\Phi,\mu) + \frac{1}{\varepsilon}\left(\|\Phi-\Psi\|^2_{L^2_\mu} + |t-s|^2\right) - \eta s.$$

Note that $[0,T]^2 \times \mathcal{C}(X,X)^2 \times \Delta(X)$ is a complete metric space when $\mathcal{C}(X,X)$ is equipped with the infinity norm and $\Delta(X)$ is equipped with the Wasserstein distance. From the Ekeland variational principle [18] applied to the lower semicontinuous function $(\Phi,\Psi) \mapsto \min_{(t,s,\mu)\in[0,T]^2\times\Delta(X)} \theta(t,s,\Phi,\Psi,\mu)$, there exists some $(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu}) \in [0,T]^2 \times \mathcal{C}(X,X)^2 \times \Delta(X)$ such that:

(E0) $\|\Phi_0 - \bar{\Psi}\|_\infty + \|\Phi_0 - \bar{\Phi}\|_\infty \leq \varepsilon$

(E1) $\theta(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu}) \leq \theta(t_0,t_0,\Phi_0,\Phi_0,\mu_0)$

(E2) for all $(t,s,\Phi,\Psi,\mu) \in [0,T]^2 \times \mathcal{C}(X,X)^2 \times \Delta(X)$:

$$\theta(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu}) \leq \theta(t,s,\Phi,\Psi,\mu) + \varepsilon(\|\Psi-\bar{\Psi}\|_\infty + \|\Phi-\bar{\Phi}\|_\infty).$$

Step 1: By (E0), we have:

$$\|\bar{\Psi}-\bar{\Phi}\|_{L^2_{\bar{\mu}}} \leq \varepsilon. \tag{28}$$

We show some estimate on $|\bar{t}-\bar{s}|$.
Indeed, applying (E2) with $(t,s,\Phi,\Psi,\bar{\mu}) = (\bar{t},\bar{t},\bar{\Phi},\bar{\Psi},\bar{\mu})$ leads to :

$$\theta(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu}) \leq \theta(\bar{t},\bar{t},\bar{\Phi},\bar{\Psi},\bar{\mu})$$

and

$$w_2(\bar{s},\bar{\Psi},\bar{\mu}) + \frac{1}{\varepsilon}|\bar{t}-\bar{s}|^2 - \eta\bar{s} \leq w_2(\bar{t},\bar{\Psi},\bar{\mu}) - \eta\bar{t}.$$

Then using (H1) :

$$\frac{1}{\varepsilon}|\bar{t}-\bar{s}|^2 \leq w_2(\bar{t},\bar{\Psi},\bar{\mu}) - w_2(\bar{s},\bar{\Psi},\bar{\mu}) + \eta|\bar{t}-\bar{s}| \leq (k+\eta)|\bar{t}-\bar{s}|$$

and we get

$$|\bar{t}-\bar{s}| \leq \varepsilon(k+\eta). \tag{29}$$

Step 2: Let us assume for a while that $\bar{s},\bar{t} \in]0,T[$ and find a contradiction.
We first show that $\frac{2}{\varepsilon}(\bar{t}-\bar{s},\bar{\Phi}-\bar{\Psi}) \in D^+_\varepsilon w_1(\bar{t},\bar{\Phi},\bar{\mu})$. Apply (E2) with $(t,s,\Phi,\Psi,\mu) = (t,\bar{s},\Phi,\bar{\Psi},\bar{\mu})$ with (t,Φ) any element of $[0,T]\times\mathcal{C}(X,X)$:

$$\theta(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu}) \leq \theta(t,\bar{s},\Phi,\bar{\Psi},\bar{\mu}) + \varepsilon\|\bar{\Phi}-\Phi\|_\infty,$$

and:

$$\begin{aligned} & -w_1(\bar{t},\bar{\Phi},\bar{\mu}) + \frac{1}{\varepsilon}\left(\|\bar{\Phi}-\bar{\Psi}\|^2_{L^2_{\bar{\mu}}} + |\bar{t}-\bar{s}|^2\right) \\ \leq \ & -w_1(t,\Phi,\bar{\mu}) + \frac{1}{\varepsilon}\left(\|\Phi-\bar{\Psi}\|^2_{L^2_{\bar{\mu}}} + |t-\bar{s}|^2\right) + \varepsilon\|\bar{\Phi}-\Phi\|_\infty. \end{aligned}$$

This yields:

$$w_1(t,\Phi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)+\frac{1}{\varepsilon}\left(\|\bar\Phi-\bar\Psi\|^2_{L^2_{\bar\mu}}-\|\Phi-\bar\Psi\|^2_{L^2_{\bar\mu}}+|\bar t-\bar s|^2-|t-\bar s|^2\right)$$
$$\leq\ \varepsilon\|\bar\Phi-\Phi\|_\infty$$

and

$$w_1(t,\Phi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)+\frac{1}{\varepsilon}\left(-\|\Phi-\bar\Phi\|^2_{L^2_{\bar\mu}}-2\langle\Phi-\bar\Phi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}-|t-\bar t|^2-2(t-\bar t)(\bar t-\bar s)\right)$$
$$\leq\ \varepsilon\left(\|\bar\Phi-\Phi\|_\infty+|\bar t-t|\right).$$

Dividing by $D:=\left(\|\Phi-\bar\Phi\|_\infty+|t-\bar t|\right)$ gives:

$$\frac{w_1(t,\Phi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)-\frac{2}{\varepsilon}\langle\Phi-\bar\Phi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}-\frac{2}{\varepsilon}(t-\bar t)(\bar t-\bar s)}{\|\Phi-\bar\Phi\|_\infty+|t-\bar t|}$$
$$\leq\ \frac{1}{\varepsilon}\frac{\|\Phi-\bar\Phi\|^2_{L^2_{\bar\mu}}+|t-\bar t|^2}{\|\Phi-\bar\Phi\|_\infty+|t-\bar t|}+\varepsilon\leq\frac{1}{\varepsilon}\frac{D^2}{D}+\varepsilon.$$

Finally:

$$\limsup_{D\to 0^+}\frac{w_1(t,\Phi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)-\frac{2}{\varepsilon}\langle\Phi-\bar\Phi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}-\frac{2}{\varepsilon}(t-\bar t)(\bar t-\bar s)}{\|\Phi-\bar\Phi\|_\infty+|t-\bar t|}\leq\varepsilon$$

so $\frac{2}{\varepsilon}(\bar t-\bar s,\bar\Phi-\bar\Psi)\in D^+_\varepsilon w_1(\bar t,\bar\Phi,\bar\mu)$. As w_1 is a subsolution of (24), we get:

$$\frac{2}{\varepsilon}(\bar t-\bar s)+\inf_{u\in U}\sup_{v\in V}\int_X\frac{2}{\varepsilon}(\bar\Phi-\bar\Psi)(x)\cdot f(\bar\Phi(x),u,v)\ d\bar\mu(x)\geq -C\varepsilon. \tag{30}$$

In the same way, we apply $(E2)$ with $(t,s,\Phi,\Psi,\mu)=(\bar t,s,\bar\Phi,\Psi,\mu)$:

$$\theta(\bar t,\bar s,\bar\Phi,\bar\Psi,\bar\mu)\leq\theta(\bar t,s,\bar\Phi,\Psi,\mu)+\varepsilon\|\Psi-\bar\Psi\|_\infty.$$

Hence for any (s,Ψ,μ) we have

$$\left\{\begin{array}{l} w_2(\bar s,\bar\Psi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)-w_2(s,\Psi,\bar\mu)+w_1(\bar t,\bar\Phi,\mu)\\ +\frac{1}{\varepsilon}\left(\|\bar\Phi-\bar\Psi\|^2_{L^2_{\bar\mu}}-\|\bar\Phi-\Psi\|^2_{L^2_{\bar\mu}}+|\bar t-\bar s|^2-|\bar t-s|^2\right)-\eta(\bar s-s)\\ \leq\varepsilon\|\Psi-\bar\Psi\|_\infty\end{array}\right. \tag{31}$$

By (H2) $\partial_- w_1(\bar t,\bar\Phi,\bar\mu)\neq\emptyset$. Fix $\xi\in\partial_- w_1(\bar t,\bar\Phi,\bar\mu)$. In view of (31) with $(s,\Psi)=(\bar s,\bar\Psi)$ we derive that $\xi\in\partial_- w_2(\bar s,\bar\Psi,\bar\mu)$. Hence

$$w_2^*(\bar s,\bar\Psi,\xi)=\langle\bar\mu,\xi\rangle-w_2(\bar s,\bar\Psi,\bar\mu)\text{ and }w_2^*(s,\Psi,\xi)\geq\langle\mu,\xi\rangle-w_2(s,\Psi,\mu),$$

$$w_1^*(\bar t,\bar\Phi,\xi)=\langle\bar\mu,\xi\rangle-w_1(\bar t,\bar\Phi,\bar\mu)\text{ and }w_1^*(\bar t,\bar\Phi,\xi)\geq\langle\mu,\xi\rangle-w_1(\bar t,\bar\Phi,\mu).$$

So in view of (31) we obtain for any (s,Ψ,μ)

$$\begin{aligned}\langle\mu,\xi\rangle - w_2(s,\Psi,\mu) - w_2^*(\bar s,\bar\Psi,\bar\xi)\\ +\frac{2}{\varepsilon}\left(\langle\Psi-\bar\Psi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}+(s-\bar s)(\bar t-\bar s)\right)+\eta(s-\bar s)\\ \le \varepsilon\|\Psi-\bar\Psi\|_\infty+\frac{1}{\varepsilon}\left(\|\Psi-\bar\Psi\|^2_{L^2_{\bar\mu}}+|s-\bar s|^2\right).\end{aligned}$$

Taking the supremum over all μ in the above inequality, this leads to

$$\begin{aligned}& w_2^*(s,\Psi,\xi)-w_2^*(\bar s,\bar\Psi,\xi)+\frac{2}{\varepsilon}\left(\langle\Psi-\bar\Psi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}+(s-\bar s)(\bar t-\bar s)\right)+\eta(s-\bar s)\\ \le\ & \varepsilon\|\Psi-\bar\Psi\|_\infty+\frac{1}{\varepsilon}\left(\|\Psi-\bar\Psi\|^2_{L^2_{\bar\mu}}+|s-\bar s|^2\right).\end{aligned}$$

Dividing by $d:=\left(\|\Psi-\bar\Psi\|_\infty+|s-\bar s|\right)$ gives:

$$\begin{aligned}& \frac{w_2^*(s,\Psi,\xi)-w_2^*(\bar s,\bar\Psi,\xi)+\frac{2}{\varepsilon}\left(\langle\Psi-\bar\Psi,\bar\Phi-\bar\Psi\rangle_{L^2_{\bar\mu}}+(s-\bar s)(\bar t-\bar s)\right)+\eta(s-\bar s)}{\|\Psi-\bar\Psi\|_\infty+|s-\bar s|}\\ \le\ & \varepsilon+\frac{1}{\varepsilon}\frac{\|\Psi-\bar\Psi\|^2_{L^2_{\bar\mu}}+|s-\bar s|^2}{d}\le\varepsilon+\frac{1}{\varepsilon}\frac{d^2}{d}.\end{aligned}$$

Taking the lim sup when d tends to 0 gives $(-\frac{2}{\varepsilon}(\bar\Phi-\bar\Psi),-\frac{2}{\varepsilon}(\bar t-\bar s)-\eta)\in D^+_{\varepsilon,\bar\mu}w_2^*(\bar s,\bar\Psi,\xi)$. Since w_2 is a dual supersolution, we deduce

$$-\frac{2}{\varepsilon}(\bar t-\bar s)-\eta-\inf_{u\in U}\sup_{v\in V}\int_X\frac{2}{\varepsilon}(\bar\Phi-\bar\Psi)(x)\cdot f(\bar\Psi(x),u,v)\ d\bar\mu(x)\ge -C\varepsilon.$$

Putting this together with (30) and using (28), we get

$$\begin{aligned}-2C\varepsilon\le-\eta+\inf_{u\in U}\sup_{v\in V}\int_X\frac{2}{\varepsilon}(\bar\Phi-\bar\Psi)(x)\cdot f(\bar\Phi(x),u,v)\ d\bar\mu(x)\\ -\inf_{u\in U}\sup_{v\in V}\int_X\frac{2}{\varepsilon}(\bar\Phi-\bar\Psi)(x)\cdot f(\bar\Psi(x),u,v)\ d\bar\mu(x)\le-\eta+2\frac{k}{\varepsilon}\|\bar\Phi-\bar\Psi\|^2_{L^2_{\bar\mu}}\le-\eta+2k\varepsilon.\end{aligned}$$

This a contradiction with (27).

Step 3 : We check that $\bar s,\bar t\neq T$. We do the proof for $\bar s=T$. Note that, by (25):

$$\theta(t_0,t_0,\Phi_0,\Phi_0,\mu_0):=w_2(t_0,\Phi_0,\mu_0)-w_1(t_0,\Phi_0,\mu_0)-\eta t_0\le\frac{-\alpha}{2},$$

which gives by (E_1):

$$\theta(\bar t,T,\bar\Phi,\bar\Psi,\bar\mu)=\theta(\bar t,\bar s,\bar\Phi,\bar\Psi,\bar\mu)\le\frac{-\alpha}{2}.$$

By definition of θ this writes as:

$$w_2(T,\bar\Psi,\bar\mu)-w_1(\bar t,\bar\Phi,\bar\mu)+\frac{1}{\varepsilon}\left(\|\bar\Psi-\bar\Phi\|^2_{L^2_{\bar\mu}}+|\bar t-\bar s|^2\right)-\eta T\le\frac{-\alpha}{2},$$

and using the k-Lipschitz property of w_1:

$$w_2(T,\bar\Psi,\bar\mu)-w_1(T,\bar\Psi,\bar\mu)-k\left(\|\bar\Psi-\bar\Phi\|_{L^2_{\bar\mu}}+|T-\bar t|\right)-\eta T\le\frac{-\alpha}{2}.$$

By (H4), this is:

$$-k\left(\|\bar\Psi-\bar\Phi\|_{L^2_{\bar\mu}}+|T-\bar t|\right)-\eta T\le\frac{-\alpha}{2}$$

and by (28) and (29):

$$\frac{\alpha}{2}\le k(k+1)\varepsilon+k\varepsilon\eta+\eta T$$

which by (27) gives

$$\frac{\alpha}{4}\le 2T\eta$$

which is a contradiction with (26).

Step 4: It remains to consider the case $\bar s$ or $\bar t=0$. It is standard to deduce from the fact that w_1 is a subsolution of (24) on $]0,T[$, that it is also a subsolution on $[0,T[$. In the same way, as w_2^* is a subsolution on $]0,T[$ of the equation (24) with $\widehat{\mathcal{H}}$ instead of $\mathcal{H}$, it is also a solution on $[0,T[$. This is enough to obtain a contradiction in the same way that in the step 2. Indeed for instance if one would suppose that $\bar s=0$ one would obtain a contradiction with the second inequality of (26). The proof is complete.

QED

5 Characterization of the Value

Proposition 4 *For any $\mu_0\in\Delta(X)$, the value functional $\mathcal{V}_r^+$ is a viscosity subsolution to the Hamilton-Jacobi equation (24).*

Proof: Let $(t_0,\Phi_0,\mu_0)\in]0,T[\times\mathcal{C}(X,X)\times\Delta(X)$ and $(p_t,p_\Phi)\in D^+_\delta\mathcal{V}_r^+(t_0,\Phi_0,\mu_0)$, we have for all $t\in]t_0,T[$, for any $\alpha\in A_c(t_0)$ and $v\in\mathcal{V}(t_0)$:

$$\begin{aligned}&\mathcal{V}_r^+(t_0,\Phi_0,\mu_0)-\mathcal{V}_r^+(t,X_t^{t_0,\Phi_0(\cdot),\alpha,v},\mu_0)+p_t(t-t_0)+\int_X(X_t^{t_0,\Phi_0(\cdot),\alpha,v}-\Phi_0)(x)\cdot p_\Phi(x)\,d\mu_0(x)\\ \ge\;&\left(\|X_t^{t_0,\Phi_0(\cdot),\alpha,v}-\Phi_0\|_\infty+|t-t_0|\right)[-\delta-\varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),\alpha,v}-\Phi_0\|_\infty+|t-t_0|\right)]\end{aligned}$$

where $\varepsilon(t)\to 0$ when $t\to 0$. As we have $X_t^{t_0,\Phi_0(x),\alpha,v}=\Phi_0(x)+\int_{t_0}^t f(X_s^{t_0,\Phi_0(x),\alpha,v},\alpha(v(s)),v(s))\,ds$, the previous expression rewrites as:

$$\begin{aligned}p_t(t-t_0)+\int_X\int_{t_0}^t f(X_s^{t_0,\Phi_0(x),\alpha,v},\alpha(v(s)),v(s))\cdot p_\Phi(x)\,dsd\mu_0(x)\\ \ge\mathcal{V}_r^+(t,X_t^{t_0,\Phi_0(\cdot),\alpha,v}\mu_0)-\mathcal{V}_r^+(t_0,\Phi_0,\mu_0)\\ +\left(\|X_t^{t_0,\Phi_0(\cdot),\alpha,v}-\Phi_0\|_\infty+|t-t_0|\right)[-\delta-\varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),\alpha,v}-\Phi_0\|_\infty+|t-t_0|\right)]\end{aligned}$$

By (2), it holds:

$$\|X_t^{t_0,\Phi_0(\cdot),\alpha,v} - \Phi_0\|_\infty \leq C|t-t_0|. \tag{32}$$

Hence

$$\begin{aligned} & p_t(t-t_0) + \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),\alpha,v}, \alpha(v(s)), v(s)) \cdot p_\Phi(x)\ dsd\mu_0(x) \\ \geq\ & \mathcal{V}_r^+(t, X_t^{t_0,\Phi_0(\cdot),\alpha,v}\mu_0) - \mathcal{V}_r^+(t_0,\Phi_0,\mu_0) - (C+1)|t-t_0|[\delta + \varepsilon\left((C+1)|t-t_0|\right)]. \end{aligned}$$

Taking the infimum in α and the supremum in v, as by Proposition 2, $\mathcal{V}_r^+$ satisfy a subdynamic principle:

$$\begin{aligned} & p_t(t-t_0) + \inf_{\alpha\in A_c(t_0)} \sup_{v\in\mathcal{V}(t_0)} \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),\alpha,v}, \alpha(v(s)), v(s)) \cdot p_\Phi(x)\ dsd\mu_0(x) \\ \geq\ & -(C+1)|t-t_0|[\delta + \varepsilon\left((C+1)|t-t_0|\right)]. \end{aligned}$$

If we reduce the infimum to the strategies α which are constant in space and time, the inequality remains:

$$\begin{aligned} & p_t(t-t_0) + \inf_{u\in U} \sup_{v\in\mathcal{V}(t_0)} \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,v}, u, v(s)) \cdot p_\Phi(x)\ dsd\mu_0(x) \\ \geq\ & -(C+1)|t-t_0|[\delta + \varepsilon\left((C+1)|t-t_0|\right)] \end{aligned}$$

Because f is bounded and Lipschitz and X is also bounded, in view of (32), we deduce that there exists a constant - denoted again by C - such that:

$$\begin{aligned} & \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,v}, u, v(s)) \cdot p_\Phi(x)\ dsd\mu_0(x) \\ \leq\ & \int_X \int_{t_0}^t f(\Phi_0(x), u, v(s)) \cdot p_\Phi(x)\ dsd\mu_0(x) + C\int_{t_0}^t |s-t_0|\ ds \\ \leq\ & (t-t_0)\left(\sup_{v\in V} \int_X f(\Phi_0(x), u, v) \cdot p_\Phi(x)\ d\mu_0(x) + C\frac{|t-t_0|}{2}\right). \end{aligned}$$

So we get:

$$\begin{aligned} & p_t(t-t_0) + (t-t_0) \inf_{u\in U} \sup_{v\in V} \left(\int_X f(\Phi_0(x), u, v) \cdot p_\Phi(x)\ d\mu_0(x) + C\frac{|t-t_0|}{2}\right) \\ \geq\ & -(C+1)|t-t_0|[\delta + \varepsilon\left((C+1)|t-t_0|\right)] \end{aligned}$$

Then, by the definition of $\mathcal{H}$, dividing by $|t-t_0|$ and letting t tend to t_0, we deduce:

$$p_t + \mathcal{H}(\mu_0, \Phi_0, p_\Phi) \geq -\delta(C+1).$$

QED

In a similar way we can prove:

Proposition 5 *For any $\mu_0 \in \Delta(X)$, the value functional $\mathcal{V}_r^-$ is a viscosity dual supersolution to the Hamilton-Jacobi equation (24).*

Proof: Let $(t_0, \Phi_0, \mu_0, \varphi) \in]0, T[\times \mathcal{C}(X,X) \times \Delta(X) \times \mathcal{C}(X)$ and $(p_t, p_\Phi) \in D^+_{\delta,\mu_0}(\mathcal{V}_r^-)^*(t_0, \Phi_0, \varphi)$, we have for all $t \in]t_0, T[$, for any $\beta \in B(t_0)$ and $u \in \mathcal{U}(t_0)$:

$$
\begin{aligned}
& (\mathcal{V}_r^-)^*(t_0, \Phi_0, \varphi) - (\mathcal{V}_r^-)^*(t, X_t^{t_0,\Phi_0(\cdot),u,\beta}, \varphi) + p_t(t - t_0) + \int_X (X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0)(x) \cdot p_\Phi(x)\, d\mu_0(x) \\
\geq\ & -\left(\|X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0\|_\infty + |t - t_0|\right)\left(\delta + \varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0\|_\infty + |t - t_0|\right)\right)
\end{aligned}
$$

where $\varepsilon(t) \to 0$ when $t \to 0$.
As we have $X_t^{t_0,\Phi_0(x),u,\beta} = \Phi_0(t_0) + \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,\beta}, u(s), \beta(u)(s))\, ds$, the previous expression rewrites as:

$$
\begin{aligned}
& p_t(t - t_0) + \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,\beta}, u(s), \beta(u)(s)) \cdot p_\Phi(x)\, ds d\mu_0(x) \\
\geq\ & (\mathcal{V}_r^-)^*(t_0, \Phi_0, \varphi) - (\mathcal{V}_r^-)^*(t, X_t^{t_0,\Phi_0(\cdot),u,\beta}, \varphi) \\
& \quad -\left(\|X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0\|_\infty + |t - t_0|\right)\left(\delta + \varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0\|_\infty + |t - t_0|\right)\right)
\end{aligned}
$$

Once again we have

$$
\|X_t^{t_0,\Phi_0(\cdot),u,\beta} - \Phi_0\|_\infty \leq C|t - t_0|.
$$

Hence

$$
\begin{aligned}
& p_t(t - t_0) + \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,\beta}, u(s), \beta(u)(s)) \cdot p_\Phi(x)\, ds d\mu_0(x) \\
\geq\ & (\mathcal{V}_r^-)^*(t_0, \Phi_0, \varphi) - (\mathcal{V}_r^-)^*(t, X_t^{t_0,\Phi_0(\cdot),u,\beta}, \varphi) - (C+1)|t - t_0|\left(\delta + \varepsilon\left((C+1)|t - t_0|\right)\right).
\end{aligned}
$$

Taking the infimum in β and the supremum in u, because by Proposition 3, $\mathcal{V}_r^-$ satisfy a dual subdynamic principle, we deduce that

$$
\begin{aligned}
& p_t(t - t_0) + \inf_{\beta \in B(t_0)} \sup_{u \in \mathcal{U}(t_0)} \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,\beta(u)}, \alpha(v(s)), v(s)) \cdot p_\Phi(x)\, ds d\mu_0(x) \\
\geq\ & -(C+1)|t - t_0|\left(\delta + \varepsilon\left((C+1)|t - t_0|\right)\right).
\end{aligned}
$$

If we reduce the infimum to the strategies β which are constant in time, the inequality remains:

$$
\begin{aligned}
p_t(t - t_0) + \inf_{v \in V} \sup_{u \in \mathcal{U}(t_0)} \int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,v}, u(s), v) \cdot p_\Phi(x)\, ds d\mu_0(x) \\
\geq -(C+1)|t - t_0|\left(\delta + \varepsilon\left((C+1)|t - t_0|\right)\right).
\end{aligned}
$$

Since f is bounded and Lipschitz and X is compact, there exists a constant - denoted again by C - such that:

$$\begin{aligned}
&\int_X \int_{t_0}^t f(X_s^{t_0,\Phi_0(x),u,v}, u(s), v) \cdot p_\Phi(x)\ ds d\mu_0(x) \\
\leq\ & \sup_{u\in U} \int_X \int_{t_0}^t f(\Phi_0(x), u, v) \cdot p_\Phi(x)\ ds d\mu_0(x) + C \int_{t_0}^t |s-t_0|\ ds \\
=\ & (t-t_0) \left(\sup_{u\in U} \int_X f(\Phi_0(x), u, v) \cdot p_\Phi(x)\ d\mu_0(x) + C\frac{|t-t_0|}{2} \right).
\end{aligned}$$

So we get:

$$\begin{aligned}
& p_t(t-t_0) + (t-t_0) \inf_{v\in V} \sup_{u\in U} \left(\int_X f(\Phi_0(x), u, v) \cdot p_\Phi(x)\ d\mu_0(x) + C\frac{|t-t_0|}{2} \right) \\
\geq\ & -(C+1)|t-t_0| \left(\delta + \varepsilon\left((C+1)|t-t_0|\right)\right).
\end{aligned}$$

Then, by the definition of $\mathcal{H}$, dividing by $|t-t_0|$ and making t tend to t_0 this yields:

$$p_t + \widehat{\mathcal{H}}(\mu_0, \Phi_0, p_\Phi) \geq -\delta(C+1).$$

QED

We state now the following main result of the article

Theorem 2 *The upper value $\mathcal{V}_r^+$ coincide with the lower value $\mathcal{V}_r^-$. Moreover the value $\mathcal{V}_r := \mathcal{V}_r^+ = \mathcal{V}_r^-$ is the unique uniformly continuous function from $[0,T]\times\mathcal{C}(X,X)\times\Delta(X)$ to $\mathbb{R}$, convex in μ, Lipschitz in (t,Φ) having a nonempty convex subdifferential at any μ which is a subsolution to (24) and a dual supersolution to (24) and which satisfies furthermore the following boundary condition*

$$\mathcal{V}_r(T,\Phi,\mu) = \int_X g(\Phi(x))d\mu(x),\ \forall \Phi \in \mathcal{C}(X,X),\ \forall \mu \in \Delta(X).$$

Proof: The boundary condition is an obvious consequence of the definition 4 of $\mathcal{V}_r^\pm$. The regularity of $\mathcal{V}_r^\pm$ has been obtained in section 2.2. By Proposition 4, the upper value $\mathcal{V}_r^+$ is a viscosity subsolution to (24) while $\mathcal{V}_r^-$ is a viscosity dual supersolution thanks to Proposition 5. From Lemma 7 functions $\mathcal{V}_r^\pm$ have nonempty subdifferential in the μ variable. Comparison Theorem 1 implies that $\mathcal{V}_r^+ = \mathcal{V}_r^-$. The proof is complete.

QED

From (13) of Lemma 2, we deduce the existence of the value of the differential game:

Corollary 1 *The differential game with asymmetric information has a value :*

$$V_r^+ = V_r^-.$$

6 Concluding remarks

- Our analysis could be also extended to a differential game with incomplete information for both players as follows. Suppose that the compact space $X \in R^N$ could be decomposed into $X = Y \times Z$. The game is played in the following way : at time t_0 an initial position $x_0 = (y_0, z_0) \in Y \times Z$ is chosen randomly according to a probability measure $\mu_0 = (\nu_0, q_0) \in \Delta(Y) \times \Delta(Z)$. The componnent y_0 of the starting position is communicated to Player I but not to Player II while z_0 is communicated to Player II but not to Player I. Both players knows ν_0 and q_0 and observe their opponents actions during the game. The Payoff is still of the form $g(x(T))$. Such a game was studied in the discrete time context in [24], for differential game with finite type information we refer the reader to [26] (cf also remark 5 in [14]).

- One can also extend our approach to the case where the payoff has the form $g(x(T)) + \int_{t_0}^{T} \ell(x(s), u(s), v(s))ds$. The main changes concern Hamiltonian and the comparison principle which are writtent in a little different way but use the same ideas that in the present paper. One can also hope to extend results of the article to differential games without Isaacs conditions. This would use the approach developed in [22].

- Surprisingly the assumption that X is compact is crucial in our approach. One can hope to treat the Hamilton Jacobi equation in $L^2(X, X)$ as in [9]. But the study of dual supersolution needs a duality between measures and functions which works well for compact sets X. This could explain why the superdifferentials in definition 5 uses a L^2 scalar product on the numerator and the $C(X, X)$ norm on the denominator, this definition of superdifferentials differs slightly from those of [9]. One could hope to extend our result to the case where X is not compact by considering instead of $\Delta(X)$ the set of probability measures on X with finite first moment which is in duality with

$$\{ \varphi \in C(X), \lim_{|x|\to\infty,\, x\in X} \frac{\varphi(x)}{1+|x|^2} = 0 \}$$

  equipped with the norm $\sup_{x\in X} \frac{|\varphi(x)|}{1+|x|^2}$ (cf e.g. [1]).

- An interesting and still widely open problem concerns the presence of signals in the incomplete informed game. We refer the reader to [31] for a specific structure of signals with finite type information.

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References

[1] M. Agueh, G. Carlier, Barycenters in the Wasserstein space. SIAM J. Math. Anal. 43 (2011), no. 2, 904-924.

[2] L. Ambrosio, W. Gangbo, Hamilton ODEs in the Wasserstein space of probability measures, Comm. Pure Appl. Math. **61** (2008), no. 1, 18–53.

[3] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[4] Aumann R (1964) Mixed and behavior strategies in infinite extensive games. Advances in Game Theory pp. 627–650 Princeton Univ. Press, Princeton, N.J.

[5] R. J. Aumann, M. B. Maschler, Repeated Games with Incomplete Information, with the Collaboration of Richard E. Stearns, MIT Press, Cambridge, MA, 1995.

[6] D. Azé, Eléments d'Analyse Convexe et Variationnelle, Ellipse, Paris 1997.

[7] Bouchitté, G.; Champion, T.; Jimenez, C. Completion of the space of measures in the Kantorovich norm. Riv. Mat. Univ. Parma (7) 4 (2005), 127-139.

[8] Buckdahn R, Cardaliaguet P, Quincampoix M (2011) Some recent aspects of differential game theory. Dyn Games Appl 1: 74-114.

[9] Cardaliaguet P, Quincampoix M (2008) Deterministic differential games under probability knowledge of initial condition. Int. Game Theory Rev. 10(1):1-16.

[10] P. Cardaliaguet, Differential games with asymmetric information. SIAM J. Control Optim. 46(3), 816-838 (2007)

[11] P. Cardaliaguet, A double obstacle problem arising in differential game theory, J. Math. Anal. Appl. 360 (1) (2009) 95-107.

[12] Cardaliaguet P, Rainer C (2012) Games with incomplete information in continuous time and for continuous types. Dyn Games Appl 2:206-227.

[13] Cardaliaguet, P, Souquière, A. A differential game with a blind player. SIAM J. Control Optim. 50 (2012), no. 4, 2090–2116.

[14] P. Cardaliaguet, C. Jimenez, M. Quincampoix, Pure and random strategies in differential game with incomplete informations, Journal of Dynamics and games 1(3): 363-375, 2014.

[15] G. Cavagnari, A. Marigonda, G. Orlandi, Hamilton-Jacobi-Bellman equation for a time-optimal control problem in the sapce of probability measures, submitted.

[16] A. Ioffe, V. Tikhomirov, Theory of Extremal problems. North-Holland. 1979.

[17] B. De Meyer, Repeated games, duality and the central limit theorem, Math. Oper. Res., 21 (1996), pp. 237-251.

[18] I. Ekeland , On the variational principle, Journal of Math. Anal. and Appl., 47 (1974), 324-353.

[19] Evans L. C, Souganidis P. E (1984) Differential games and representation formulas for solutions of Hamilton Jacobi Isaacs equations. Indiana Univ. Math. J. 33: 773–797.

[20] W. Gangbo, T. Nguyen, A. Tudorescu. Hamilton-Jacobi equations equations in the Wasserstein space. Method Appl. Anal. 15 (2008) .no. 4: 155-183.

[21] Hanin, Leonid G. An extension of the Kantorovich norm. Monge AmpÃ¨re equation: applications to geometry and optimization (Deerfield Beach, FL, 1997), 113-130, Contemp. Math., 226, Amer. Math. Soc., Providence, RI, 1999.

[22] Jimenez, Chloe; Quincampoix, Marc; Xu, Yuhong Differential games with incomplete information on a continuum of initial positions and without Isaacs condition. Dyn. Games Appl. 6 (2016), no. 1, 82-96.

[23] Lasry, Jean-Michel; Lions, Pierre-Louis Mean field games. Jpn. J. Math. 2 (2007), no. 1, 229–260.

[24] Mertens J.-F, Zamir S (1971) The value of two-person zero-sum repeated games with lack of information on both sides. Internat. J. Game Theory 1: 39–64.

[25] Mertens, J.-F.; Sorin, S.; Zamir, S. Repeated games. Econometric Society Monographs. Cambridge University Press, New York, 2015.

[26] Oliu-Barton, Miquel (2015) Differential games with asymmetric and correlated information. Dyn. Games Appl. 5 , no. 3, 378-396.

[27] Santambrogio, Filippo, Optimal transport for applied mathematicians. Calculus of variations, PDEs, and modeling. Progress in Nonlinear Differential Equations and their Applications, 87. Birkhäuser/Springer, Cham, 2015.

[28] Souquière, A. (2009) Approximation and representation of the value for some differential games with asymmetric information. Int. J. Game Theory 39, 4, 699-722.

[29] Schmeidler D (1973) Equilibrium points of nonatomic games. Journal of Statistical Physics, 4: 295-300.

[30] Villani C (2003) Topics in optimal transportation. Graduate studies in Mathematics, Vol.58, AMS.

[31] Wu, Xiaochi (2017) Existence of value for differential games with incomplete information and signals on initial states and payoffs. J. Math. Anal. Appl. 446 , no. 2, 1196-1218.