ANALYSIS OF A DYNAMIC PEELING TEST WITH SPEED-DEPENDENT TOUGHNESS

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Abstract. We analyse a one-dimensional model of dynamic debonding for a thin film, where the local toughness of the glue between the film and the substrate also depends on the debonding speed. The wave equation on the debonded region is strongly coupled with Griffith's criterion for the evolution of the debonding front. We provide an existence and uniqueness result and find explicitly the solution in some concrete examples. We study the limit of solutions as inertia tends to zero, observing phases of unstable propagation, as well as time discontinuities, even though the toughness diverges at a limiting debonding speed.

Keywords: Dynamic debonding; Wave equation in time-dependent domains; Quasistatic limit; Griffith's criterion; Dynamic fracture; Thin films.
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Introduction

In models of crack propagation in load-bearing structures, mathematical analysis is central in showing well-posedness and approximation properties. Under the assumption that inertial effects are negligible, quasistatic evolution was extensively studied, see e.g. [12, 9, 2, 11, 3, 1, 6], proving existence of solutions in agreement with the theory of rate-independent processes [26]. Fewer results were given in dynamic models, which are more intricate due to internal oscillations of the material: in the sharp crack case [28, 4, 8, 5], one needs to prescribe a priori the crack path, as well as in delamination models [29, 31, 30]. Other results deal with phase-field approximations, see e.g. [18].

In the present paper we study a model of dynamic debonding, which can be regarded as a simplified model of fracture as already observed in [13, Section 7.4]. More precisely, we consider a thin film peeled from a rigid substrate where it is initially glued. Assuming that the process only depends on one space variable and using a linearisation, we reduce to a PDE system consisting of:

- the wave equation, satisfied in a time-dependent interval parametrising the debonded part of the film;
- a first-order flow rule, namely Griffith's criterion, dictating the evolution of the debonding front.

The latter law is a threshold condition stating that the domain of the wave equation is non-decreasing and may only grow if there is an equality between the dynamic energy release rate and the local toughness of the glue. The strong coupling of the momentum equation with a propagation criterion is typical of dynamic fracture; the peeling model analysed here is one of the few cases where a complete mathematical solution can be found.

Previous studies of such one-dimensional problem concerned the analysis of particular cases [10, 20], existence and uniqueness of solutions [7], and their quasistatic limit [19]. All of these papers assume that the local toughness only depends on the position in the reference configuration,

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the most general case being a piecewise Lipschitz function with a finite number of discontinuities (modelling composites). In the present paper we extend such results to a toughness also depending on the debonding speed, i.e., on the time derivative of the position of the debonding front.

In fracture models, toughness is insensitive to crack speed only for low speeds, while it increases with the crack speed: this is due to inertial effects [25, Section 5.1.5]. In metals, the local toughness features a sharp upturn at a limiting crack speed [23]. (Indeed, the crack speed is less than the speed of sound in the elastic domain.) Notice that our results are compatible with assuming that the toughness blows up as the crack speed approaches the speed of sound. Thus we give a theoretical validation of the aforementioned mechanical models, by means of an existence and uniqueness theorem and of some examples where solutions are explicitly computed.

On the other hand, the toughness/speed relation may not be monotonic: this occurs e.g. in polymers (PMMA, epoxy, rubber) and in peeling of polymeric adhesives [32]. Moreover, the presence of regions where toughness decreases is observed in rate-dependent solids also when inertia is negligible [17]. We are able to treat non-monotonic dependence of toughness with respect to speed, provided its slope is bounded from below; this is compatible with experimental laws. Oscillations in the toughness/speed curve are responsible of phenomena of arrest/fast propagation and rule out steady growth. To account for this, one may include a dependence on crack acceleration [24, 32, 14] which is not considered in our model and may be a further development.

We refer to Section 1 for the statement of the problem and of our assumptions. It is possible to see that Griffith's criterion can be decoupled from the wave equation [7]; hence, the equality between the dynamic energy release rate and the local toughness κ reduces to an ordinary differential equation for the debonding front $t \mapsto \ell(t)$, of the form

$$\dot{\ell}(t) = F(t, \ell(t), \kappa(\ell(t), \dot{\ell}(t))),$$

cf. (1.11). Solving this ordinary differential equation is the first difficulty of the present paper; in fact, it is not expressed in normal form, since the local toughness may explicitly depend on the debonding speed $\dot{\ell}$. In Section 2 we prove that $\mu \mapsto F(t, \ell, \kappa(\ell, \mu))$ is invertible for fixed t, ℓ; in this step we use the assumption that the toughness/speed curve has slope bounded from below (Lemma 2.1). We then need some careful estimates on the inverse of F, based on the assumption that the toughness has a Lipschitz dependence on ℓ (Lemma 2.2). They finally allow us to find a unique evolution $t \mapsto \ell(t)$ satisfying Griffith's criterion, by applying classical results on ordinary differential equations (Theorem 2.3).

Once the existence and uniqueness result is established, in Section 3 we study the quasistatic limit of debonding evolutions, i.e., the limit of the system for small loading speed. Up to a time rescaling, this is equivalent to assume that inertia tends to zero and the speed of sound tends to infinity ("vanishing inertia" limit). Therefore, one may expect that dynamic solutions converge to a rate-independent evolution, as in damage models [31, 21]. However, such a convergence may fail even for potential-type equations in finite dimension [27]. In fact, some counterexamples [20, 19] show that in general the limit of dynamic debonding evolutions for slow loading does not satisfy Griffith's criterion in its quasistatic version, when the local toughness is independent of the debonding speed.

In this paper, we first consider a local toughness given by

$$\kappa(\ell, \dot{\ell}) = \tilde{\kappa} + \gamma\dot{\ell},$$

for $\tilde{\kappa}, \gamma$ positive constants. In this case, Griffith's propagation condition has the same form of the corresponding equation in inertia-free fracture models with a viscous regularisation [16, 22]. In the dynamic case, one may ask if the extra term $\gamma\dot{\ell}$ favours convergence of the flow rule for slow loading. We show that the answer is negative. Indeed, starting from certain initial conditions far from equilibrium, the dynamic solutions present alternation of phases of arrest and propagation; in the limit, they converge to a slow unstable transition that can be determined analytically and does not fulfil any notion of rate-independent evolution (see Example 1). The same phenomenon occurs if the toughness satisfies the physical assumption that

$$\kappa(\ell,\mu) \to +\infty, \quad \text{as } \mu \to c,$$

where c is the speed of sound.

The latter assumption penalises high debonding speed in dynamic evolutions. One may ask if it also prevents brutal propagations, i.e., time discontinuities, in the quasistatic limit. Once again, the answer is negative: this is shown in Example 2, for a local toughness with a discontinuous dependence on the position. We observe that a sudden decrease in toughness produces fast propagations, where the debonding speed is way smaller, but of the same order, than the speed of sound. As a consequence, the quasistatic limit features a jump of the debonding front in time.

Our results show that a dependence of the local toughness upon the debonding speed may be included in dynamic debonding models, under weak assumptions. However, it provides no regularising effects on the quasistatic limit. Understanding conditions that guarantee rate-independence for vanishing inertia will be matter of further investigation.

1. Dynamic peeling

We now describe the peeling model under consideration. The reference configuration of the film is the horizontal half plane $\{(x, y, z) : x \geq 0,\ z = 0\}$, which coincides with the substrate where the film is initially attached. The deformed configuration is given by $(x, y) \mapsto (x + h(t, x), y, u(t, x))$, where h, u are two functions; i.e., the displacement is $(h(t, x), 0, u(t, x))$. The second component is fixed, thus the parametrisation reduces to one space dimension. See Figure 1.

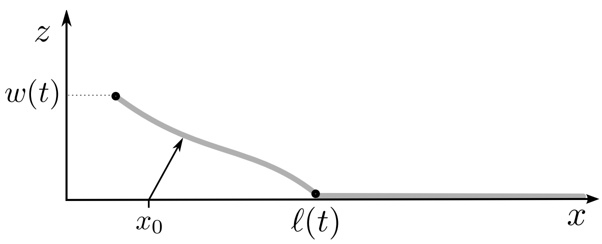

Figure 1. The deformation of the debonded film is represented by its section, the curve $x \mapsto (x + h(t, x), u(t, x))$. The vector applied to the point x_0 in figure is the displacement $(h(t, x_0), u(t, x_0))$.

We assume that the film is perfectly flexible and inextensible. The bonded part of the film is the half line $\{(x, y, z) : x \geq \ell(t)\,,\ z = 0\}$, where $\ell(t)$ is a non-decreasing function representing the debonding front, with $\ell_0 := \ell(0) > 0$; i.e., $h(t, x) = u(t, x) = 0$ for $x \geq \ell(t)$. At the endpoint $x = 0$ there is a time-dependent boundary condition on the vertical displacement $u(t, 0) = w(t)$; the tension is fixed in such a way that the speed of sound in the debonded part of the film is constant (normalised to one).

By linear approximation and inextensibility, the horizontal displacement h is known through the vertical part u:

$$h(t, x) = \frac{1}{2} \int_x^{+\infty} u_x(t, \xi)^2 \, \mathrm{d}\xi.$$

In its turn, u solves the problem

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t > 0\,,\, 0 < x < \ell(t), \tag{1.1a}$$

$$u(t,0) = w(t), \quad t > 0, \tag{1.1b}$$

$$u(t,\ell(t)) = 0, \quad t > 0, \tag{1.1c}$$

with initial conditions

$$u(0,x) = u_0(x), \quad 0 < x < \ell_0, \tag{1.1d}$$

$$u_t(0,x) = u_1(x), \quad 0 < x < \ell_0. \tag{1.1e}$$

Here we require that

$$u_0 \in C^{0,1}([0,\ell_0]), \quad u_1 \in L^\infty(0,\ell_0), \quad \text{and} \quad w \in \widetilde{C}^{0,1}(0,+\infty), \tag{1.2a}$$

where

$$\widetilde{C}^{0,1}(0,+\infty) := \{f \in C^0([0,+\infty)) : f \in C^{0,1}([0,T]) \text{ for every } T > 0\}.$$

We assume the following compatibility conditions:

$$u_0(0) = w(0) \quad \text{ and } u_0(\ell_0) = 0. \tag{1.2b}$$

1.1. **The wave equation.** In order to make precise the notion of solution of problem (1.1), we assume for the moment that the evolution of the debonding front is prescribed. Fix $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ Lipschitz and such that

$$0 \le \dot\ell(t) < 1, \text{ for a.e. } t > 0, \tag{1.3a}$$

$$\ell(0) = \ell_0. \tag{1.3b}$$

Moreover, we set

$$\Omega := \{(t,x) \in (0,+\infty){\times}(0,=\infty) : 0 < x < \ell(t)\},$$

and, given any $T > 0$,

$$\Omega_T := \Omega \cap \{t < T\}.$$

Let

$$\widetilde{H}^1(\Omega) := \{u \in H^1_{\rm loc}(\Omega) : u \in H^1(\Omega_T), \text{ for every } T > 0\},$$

$$\widetilde{C}^{0,1}(\Omega) := \{u \in C^0(\Omega) : u \in C^{0,1}(\Omega_T) \text{ for every } T > 0\}.$$

Definition 1.1. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. $u \in H^1(\Omega_T)$) is a solution to* (1.1) *if $u_{tt} - u_{xx} = 0$ holds in the sense of distributions in Ω (resp. in Ω_T), the boundary conditions* (1.1b) *&* (1.1c) *are intended in the sense of traces and the initial conditions* (1.1d) *&* (1.1e) *are also satisfied in the sense of traces of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.*

To give precise meaning to condition (1.1e) we notice that $u_x \in L^2(0,T;L^2(0,\ell_0))$, thus $u_{tt} = u_{xx} \in L^2(0,T;H^{-1}(0,\ell_0))$, therefore $u_t \in H^1(0,T;H^{-1}(0,\ell_0)) \subset C^0([0,T];H^{-1}(0,\ell_0))$. The following result was proved in [7, Section 1].

Proposition 1.2. *Assume* (1.2) *and* (1.3)*. Then, there exists a unique solution $u \in H^1(\Omega)$ to problem* (1.1)*, according to Definition 1.1. Moreover, $u \in \widetilde{C}^{0,1}(\Omega)$ and is expressed through the formula*

$$u(t,x) = w(t{+}x) - f(t{+}x) + f(t{-}x),$$

where $f \in \widetilde{C}^{0,1}(-\ell_0,+\infty)$ is determined by

$$w(t+\ell(t)) - f(t+\ell(t)) + f(t-\ell(t)) = 0, \quad \textit{for every } t > 0, \tag{1.4}$$

and

$$f(s) = w(s) - \frac{1}{2}u_0(s) - \frac{1}{2}\int_0^s u_1(x)\,\mathrm{d}x - w(0) + \frac{1}{2}u_0(0), \quad \textit{for every } s \in [0, \ell_0], \tag{1.5a}$$

$$f(s) = \frac{1}{2}u_0(-s) - \frac{1}{2}\int_0^{-s} u_1(x)\,\mathrm{d}x - \frac{1}{2}u_0(0), \quad \textit{for every } s \in (-\ell_0, 0]. \tag{1.5b}$$

1.2. **Griffith's criterion.** We now introduce the flow rule to determine the evolution of the debonding front $t \mapsto \ell(t)$ when it is unknown. We start from the internal energy

$$\mathcal{E}(t; \ell, w) = \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2\,\mathrm{d}x, \tag{1.6}$$

which is well defined for every ℓ and w thanks to Proposition 1.2. In terms of the function of one variable f, by (1.6) one obtains

$$\mathcal{E}(t; \ell, w) = \int_{t-\ell(t)}^{t} \dot{f}(s)^2\,\mathrm{d}s + \int_t^{t+\ell(t)} [\dot{w}(s) - \dot{f}(s)]^2\,\mathrm{d}s. \tag{1.7}$$

This is to be compared with the energy dissipated in debonding in the time interval $(0,t)$, given by

$$\int_0^t \kappa(\ell(s), \dot{\ell}(s))\,\mathrm{d}s,$$

where κ is the local toughness of the glue between the film and the substrate.

We assume that $\kappa(x,\mu)$ is a measurable function of the position x in the reference configuration and of the debonding speed μ,

$$\kappa\colon [0,+\infty)\times[0,1) \to [c_1, +\infty), \tag{1.8a}$$

where $c_1 > 0$. For every $\mu \in [0,1)$, we require that $\kappa(\cdot,\mu)$ is piecewise Lipschitz with a finite number of discontinuities at points $\ell_1 < \cdots < \ell_N$, that it has finite left- and right-sided limits at $\ell_1, \ldots, \ell_N$, and that it satisfies

$$|\kappa(x_1,\mu) - \kappa(x_2,\mu)| \le L|x_1 - x_2|(\kappa(x_1,\mu) + \kappa(x_2,\mu)) \quad \text{for } x_1, x_2 \in (\ell_j, \ell_{j+1}),\ j \ge 0. \tag{1.8b}$$

Moreover, for every $x \ge \ell_0$ and $\mu_1, \mu_2 \in [0,1)$ we assume

$$\frac{\kappa(x,\mu_2) - \kappa(x,\mu_1)}{\mu_2 - \mu_1} > -c_3\frac{(\sqrt{\kappa(x,\mu_2)} + \sqrt{\kappa(x,\mu_1)})^2}{4}, \tag{1.8c}$$

where $c_3 < 2$. Notice that this condition is automatically fulfilled when κ is non-decreasing with respect to μ; in general, it requires a bound from below on its slope. It will be used in Lemma (2.1).

Next, one defines the dynamic energy release rate $G_{\dot{\ell}(t)}(t)$ as the opposite of a (sort of) partial derivative of $\mathcal{E}$ with respect to ℓ. Given ℓ as in (1.3) and $\alpha \in (0,1)$, we consider extensions $\lambda \in C^{0,1}([0,+\infty))$ such that $\lambda(t) = \ell(t)$ for every $0 \le t \le t_0$, $\dot{\lambda} < 1$ for a.e. $t > 0$, and

$$\frac{1}{h}\int_{t_0}^{t_0+h} \left|\dot{\lambda}(t) - \alpha\right|\,\mathrm{d}t \to 0, \qquad \text{as } h \to 0^+.$$

We freeze the external loading at time t_0 by setting

$$z(t) = \begin{cases} w(t), & t \le t_0, \\ w(t_0), & t > t_0. \end{cases}$$

The dynamic energy release rate $G_\alpha(t_0)$ at time t_0 corresponding to a debonding speed $0 < \alpha < 1$ is

$$G_\alpha(t_0) := \lim_{t \to t_0^+} \frac{\mathcal{E}(t_0; \lambda, z) - \mathcal{E}(t; \lambda, z)}{(t - t_0)\alpha}.$$

The existence of such limit is proved in [7, Section 2]. Moreover, by (1.7) we obtain the following formula:

$$G_\alpha(t) = 2\,\frac{1-\alpha}{1+\alpha}\dot{f}(t-\ell(t))^2. \tag{1.9}$$

In particular, G_α depends on λ only through α. For $\alpha = 0$, we set

$$G_0(t) := 2\dot{f}(t-\ell(t))^2.$$

The dynamic energy release rate is compared with the energy dissipated in an infinitesimal propagation of the debonding, that is the local toughness (1.8a). Griffith's criterion reads as follows:

$$\begin{cases} \dot\ell(t) \geq 0, \\ G_{\dot\ell(t)}(t) \leq \kappa(\ell(t), \dot\ell(t)), \\ \Big[G_{\dot\ell(t)}(t) - \kappa(\ell(t), \dot\ell(t))\Big]\,\dot\ell(t) = 0. \end{cases} \tag{1.10}$$

Notice that the third equation says that the product of two nonnegative quantities is zero; thus, the process is activated ($\dot\ell \neq 0$) only when the energy release rate is critical (equal to the toughness). Using (1.9), this flow rule is rephrased in terms of a Cauchy problem for the evolution of the debonding front $t \mapsto \ell(t)$. We obtain indeed the following formulation, equivalent to (1.10):

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t), \dot\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t), \dot\ell(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{1.11}$$

Our aim is then to solve the coupled problem (1.1) & (1.11), where we use the notion of solution given in Definition 1.1.

2. Existence and uniqueness

Since the local toughness depends also on $\dot\ell(t)$, our main difficulty is that the ordinary differential equation in (1.11) is not expressed in normal form. To overcome this difficulty, we introduce the variable $z(t) := t - \ell(t)$ and we consider the function

$$\Phi\colon [0, +\infty) \times [-\ell_0, \ell_0] \times [0, +\infty) \to \mathbb{R}$$

defined (for every t, μ and a.e. z) by

$$\Phi(t, z, \mu) := \begin{cases} \mu - \dfrac{2\dot f(z)^2 - \kappa(t-z, \mu)}{2\dot f(z)^2 + \kappa(t-z, \mu)}, & \text{if } 2\dot f(z)^2 \geq \kappa(t-z, \mu), \\ \mu, & \text{if } 2\dot f(z)^2 < \kappa(t-z, \mu). \end{cases} \tag{2.1}$$

Our strategy is then to prove that $\mu \mapsto \Phi(t, z, \mu)$ is invertible for fixed t, z. This will ensure that the Cauchy problem can be recast in normal form.

Following [7], we notice that in the triangle $\{(t, x) : x \in [0, \ell_0],\ |t| \leq \ell_0 - x\}$ the solution u of the wave equation (1.1) is independent of ℓ. Therefore, starting from (1.2), by Proposition 1.2 we obtain the one-dimensional function f in the interval $[-\ell_0, \ell_0]$. We then want to solve (1.11) as long as $z(t) = t - \ell(t) \in [-\ell_0, \ell_0]$, knowing that $f \in C^{0,1}([-\ell_0, \ell_0])$.

Lemma 2.1. *Let κ be as in (1.8), $f \in C^{0,1}([-\ell_0, \ell_0])$, and Φ as in (2.1). Then,*

$$\frac{\Phi(t,z,\mu_2) - \Phi(t,z,\mu_1)}{\mu_2 - \mu_1} > 1 - \frac{c_3}{2},$$

for every $t > 0$, a.e. $z > -\ell_0$, and every $0 < \mu_1 \leq \mu_2$, where c_3 is given in (1.8c).

Proof. Let $t > 0$ and $z > -\ell_0$. We first observe that if $\Phi(t,z,\mu) = \mu$ as in the second line of (2.1), then the thesis trivially holds. Next we prove it when Φ is given by the first line. This leads to the conclusion, since Φ is the minimum of two functions whose difference quotients are controlled from below.

We can conclude by showing that Φ is increasing also when it is equal to $\mu - \frac{2\dot f(z)^2 - \kappa(t-z,\mu)}{2\dot f(z)^2 + \kappa(t-z,\mu)}$. Let $0 < \mu_1 \leq \mu_2$ and assume that $\Phi(t,z,\mu_i) = \mu_i - \frac{2\dot f(z)^2 - \kappa(t-z,\mu_i)}{2\dot f(z)^2 + \kappa(t-z,\mu_i)}$ for $i = 1, 2$. Then,

$$\begin{aligned}
\frac{\Phi(t,z,\mu_2) - \Phi(t,z,\mu_1)}{\mu_2 - \mu_1} &= 1 - \frac{1}{\mu_2 - \mu_1}\frac{2\dot f(z)^2 - \kappa(t-z,\mu_2)}{2\dot f(z)^2 + \kappa(t-z,\mu_2)} + \frac{1}{\mu_2 - \mu_1}\frac{2\dot f(z)^2 - \kappa(t-z,\mu_1)}{2\dot f(z)^2 + \kappa(t-z,\mu_1)} \\
&= 1 + \frac{4\dot f(z)^2[\kappa(t-z,\mu_2) - \kappa(t-z,\mu_1)]}{(2\dot f(z)^2 + \kappa(t-z,\mu_1))(2\dot f(z)^2 + \kappa(t-z,\mu_2))(\mu_2 - \mu_1)} \\
&> 1 - \frac{c_3}{2}\frac{2\dot f(z)^2(\sqrt{\kappa(t-z,\mu_1)} + \sqrt{\kappa(t-z,\mu_2)})^2}{(2\dot f(z)^2 + \kappa(t-z,\mu_1))(2\dot f(z)^2 + \kappa(t-z,\mu_2))}.
\end{aligned} \tag{2.2}$$

This holds for every $t > 0$ and a.e. $z > -\ell_0$. Notice that in the last line we used (1.8c). Moreover, it is easy to see that

$$\frac{\alpha}{(\alpha + \kappa(t-z,\mu_1))(\alpha + \kappa(t-z,\mu_2))} \leq \frac{1}{(\sqrt{\kappa(t-z,\mu_1)} + \sqrt{\kappa(t-z,\mu_2)})^2}$$

for every $\alpha \geq 0$. Therefore, we can continue (2.2) and deduce that

$$\frac{\Phi(t,z,\mu_2) - \Phi(t,z,\mu_1)}{\mu_2 - \mu_1} > 1 - \frac{c_3}{2},$$

where $c_3 < 2$ as stated in (1.8c). □

By Lemma 2.1, the function $\mu \mapsto \Phi(t,z,\mu)$, that maps $[0,+\infty)$ into itself, is globally invertible for every $t \geq 0$ and a.e. $z \in [-\ell_0, \ell_0]$. Let then $\Psi\colon [0,+\infty) \times [-\ell_0,\ell_0] \times [0,+\infty) \to [0,+\infty)$ be the function such that, given $\sigma \in [0,+\infty)$,

$$\Phi(t,z,\Psi(t,z,\sigma)) = \sigma \quad \text{and} \quad \Psi(t,z,\Phi(t,z,\mu)) = \mu,$$

for every $t \geq 0$ and a.e. $z \in [-\ell_0, \ell_0]$. We can thus rephrase problem (1.11) as

$$\begin{cases} \dot\ell(t) = \Psi(t, t-\ell, 0), & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0, \end{cases}$$

now expressed in normal form. Following [7, Theorem 3.5], it is convenient to use the equivalent form

$$\begin{cases} 1 - \dot z(t) = \Psi(t, z, 0), & \text{for a.e. } t > 0, \\ z(0) = -\ell_0. \end{cases} \tag{2.3}$$

We now prove that $t \mapsto \Psi(t,z,0)$ is Lipschitz for fixed z.

Lemma 2.2. *Consider Φ and Ψ as above. Let κ be as in (1.8). Then, there exists $C > 0$ such that*

$$|\Psi(t_2,z,\sigma) - \Psi(t_1,z,\sigma)| \leq C|t_2 - t_1|,$$

for every $t_1, t_2 > 0$ and a.e. $z > -\ell_0$ such that $t_1, t_2 \in (\ell_j + z, \ell_{j+1} + z)$ for some $j \geq 0$, and for every $\sigma > 0$.

Proof. We start from showing that Φ is Lipschitz in t. Let $t_1, t_2 > 0$, $z \in [-\ell_0, \ell_0]$, and $\mu > 0$ as in the statement, such that $\Phi(t_1, z, \mu)$ and $\Phi(t_2, z, \mu)$ are defined. Then, we consider $x_1, x_2 > \ell_0$ such that $x_i = t_i - z$ for $i = 1, 2$. We have

$$
\begin{aligned}
\Phi(t_1, z, \mu) - \Phi(t_2, z, \mu) &= \frac{2\dot f(z)^2 - \kappa(x_1, \mu)}{2\dot f(z)^2 + \kappa(x_1, \mu)} - \frac{2\dot f(z)^2 - \kappa(x_2, \mu)}{2\dot f(z)^2 + \kappa(x_2, \mu)} \\
&= \frac{2\dot f(z)^2 - \kappa(x_1, \mu)}{2\dot f(z)^2 + \kappa(x_1, \mu)} - \frac{2\dot f(z)^2 - \kappa(x_2, \mu)}{2\dot f(z)^2 + \kappa(x_1, \mu)} + \frac{2\dot f(z)^2 - \kappa(x_2, \mu)}{2\dot f(z)^2 + \kappa(x_1, \mu)} - \frac{2\dot f(z)^2 - \kappa(x_2, \mu)}{2\dot f(z)^2 + \kappa(x_2, \mu)} \\
&= \frac{\kappa(x_2, \mu) - \kappa(x_1, \mu)}{2\dot f(z)^2 + \kappa(x_1, \mu)} + (\kappa(x_2, \mu) - \kappa(x_1, \mu)) \frac{2\dot f(z)^2 - \kappa(x_2, \mu)}{(2\dot f(z)^2 + \kappa(x_1, \mu))(2\dot f(z)^2 + \kappa(x_2, \mu))} \\
&= (\kappa(x_2, \mu) - \kappa(x_1, \mu)) \frac{4\dot f(z)^2}{(2\dot f(z)^2 + \kappa(x_1, \mu))(2\dot f(z)^2 + \kappa(x_2, \mu))}.
\end{aligned}
$$

This implies that, by (1.8b),

$$
\begin{aligned}
&|\Phi(t_1, z, \mu) - \Phi(t_2, z, \mu)| \\
&\leq 4L|x_1 - x_2| \dot f(z)^2 \frac{\kappa(x_1, \mu) + \kappa(x_2, \mu)}{(2\dot f(z)^2 + \kappa(x_1, \mu))(2\dot f(z)^2 + \kappa(x_2, \mu))} \\
&\leq 4L|x_1 - x_2| = 4L|t_1 - t_2|,
\end{aligned}
\tag{2.4}
$$

where in the last line we used the fact that

$$
(\kappa(x_1, \mu) + \kappa(x_2, \mu)) \dot f(z)^2 \leq (2\dot f(z)^2 + \kappa(x_1, \mu))(2\dot f(z)^2 + \kappa(x_2, \mu)).
$$

We now notice that, for every $\sigma_1, \sigma_2 > 0$, we have

$$
\begin{aligned}
1 &= \frac{\Phi(t, z, \Psi(t, z, \sigma_2)) - \Phi(t, z, \Psi(t, z, \sigma_1))}{\sigma_2 - \sigma_1} \\
&= \frac{\Phi(t, z, \Psi(t, z, \sigma_2)) - \Phi(t, z, \Psi(t, z, \sigma_1))}{\Psi(t, z, \sigma_2) - \Psi(t, z, \sigma_1)} \frac{\Psi(t, z, \sigma_2) - \Psi(t, z, \sigma_1)}{\sigma_2 - \sigma_1}.
\end{aligned}
$$

Therefore, by Lemma 2.1,

$$
\frac{|\Psi(t, z, \sigma_2) - \Psi(t, z, \sigma_1)|}{\sigma_2 - \sigma_1} < \frac{1}{1 - \frac{c_3}{2}}. \tag{2.5}
$$

Moreover, for every $\mu > 0$, we have

$$
\begin{aligned}
0 &= \frac{\Psi(t_2, z, \Phi(t_2, z, \mu)) - \Psi(t_1, z, \Phi(t_1, z, \mu))}{t_2 - t_1} \\
&= \frac{\Psi(t_2, z, \Phi(t_2, z, \mu)) - \Psi(t_1, z, \Phi(t_2, z, \mu))}{t_2 - t_1} \\
&\quad + \frac{\Psi(t_1, z, \Phi(t_2, z, \mu)) - \Psi(t_1, z, \Phi(t_1, z, \mu))}{\Phi(t_2, z, \mu) - \Phi(t_1, z, \mu)} \frac{\Phi(t_2, z, \mu) - \Phi(t_1, z, \mu)}{t_2 - t_1}.
\end{aligned}
$$

Finally, for every $\sigma > 0$ there exists $\mu > 0$ such that $\sigma = \Phi(t_2, z, \mu)$ (by invertibility of $\mu \mapsto \Phi(t_2, z, \mu)$) and, by (2.4) and (2.5),

$$
|\Psi(t_2, z, \sigma) - \Psi(t_1, z, \sigma)| \leq \frac{4L}{1 - \frac{c_3}{2}} |t_2 - t_1|.
$$

This concludes the proof. $\square$

The following result shows existence and uniqueness of a pair (u,ℓ) solving the coupled problem (1.1) & (1.11). This generalises [7, Theorem 3.5] to the case of a speed-dependent toughness.

Theorem 2.3. *Assume* (1.8) *and* (1.2). *Then, there exists a unique pair* $(u,\ell) \in \widetilde{H}^1(\Omega) \times \widetilde{C}^{0,1}(0,+\infty)$ *solving* (1.1)*&*(1.11). *Moreover,* $u \in \widetilde{C}^{0,1}(\Omega)$ *and for every* $T>0$ *there exists* $L_T<1$ *such that* $\dot\ell \le L_T$.

Proof. We first consider the case where κ is continuous. We have to construct a function f satisfying (1.4) and a function ℓ satisfying (1.11). By (1.5) we are provided f in the interval $[-\ell_0,\ell_0]$ and we know that f is Lipschitz. Next we solve the Cauchy problem (2.3) as long as $t-\ell(t)\in[-\ell_0,\ell_0]$. We see that Ψ is measurable in z, since

$$\{z>-\ell_0 : \Psi(t,z,\sigma)<\mu\} = \{\sigma>0 : \sigma<\Phi(t,z,\mu)\}, \quad \text{for every } t,\mu>0,$$

where Φ is in its turn measurable since $\dot f \in L^\infty(-\ell_0,\ell_0)$ and $\kappa>c_1$ is piecewise Lipschitz. Moreover, by Lemma 2.2, $t\mapsto\Psi(t,z,0)$ is locally Lipschitz for a.e. $z\in(-\ell_0,\ell_0)$. We now notice that there exists $0<c<1$ such that

$$\Psi(t,z,0)\in[0,1-c].$$

Indeed, starting from $\Phi(t,z,\Psi(t,z,0))=0$, we find that

$$\Psi(t,z,0)=\frac{2\dot f(z)^2-\kappa(t-z,\Psi(t,z,0))}{2\dot f(z)^2+\kappa(t-z,\Psi(t,z,0))}\vee 0.$$

Therefore, every solution to (2.3) must satisfy $\dot z(t)>0$ for a.e. $t>0$ and it is thus invertible. The function $z\mapsto t(z)$ solves the problem

$$\begin{cases} \dot t(z) = \dfrac{1}{1-\Psi(t,z,0)}, & \text{for a.e. } z>-\ell_0,\\ t(-\ell_0)=0. \end{cases} \tag{2.6}$$

We observe that $0\le \dot t(z)\le \frac{1}{c}$, $\Psi(t,z,0)$ is Lipschitz in t uniformly in z, and it is measurable in z; then we can apply classical results on ordinary differential equations (see, e.g., [15, Theorem 5.3]) to get a unique solution $z\mapsto t(z)$ to (2.6). Next, z is found by inverting the function $t(z)$ and finally $\ell(t)=t-z(t)$ is the unique solution to (1.11) up to time $t(\ell_0)$, satisfying $\dot\ell\le L_T$. Next we employ (1.4) to extend f to $(\ell_0, t(\ell_0)+\ell(t(\ell_0))]$, so the ordinary differential equation can be solved in this interval, hence ℓ and f are further extended. Iterating this argument we extend the solution ℓ to $[0,+\infty)$.

In the case that κ has a finite number of discontinuities $\ell_1,\dots,\ell_N$, we may apply the previous argument to solve (2.6) as long as $t(z)-z<\ell_1$. If there is z_1 such that $t(z_1)=\ell_1+z_1$, we extend the solution for $z\ge z_1$ by solving the Cauchy problem with initial datum $t(z_1)=\ell_1+z_1$ as long as $t(z)-z<\ell_2$, recalling the monotonicity of $z\mapsto t(z)-z$. Iterating this argument allows us to conclude. ☐

3. Quasistatic limit

Following [19] we now study the limit behaviour of the system when the external loading w is quasistatic, namely when we replace $w(t)$ with $w(\varepsilon t)$ and $\varepsilon>0$ is a small parameter, so that the speed of the vertical displacement is very slow.

We call $(u_\varepsilon(t,x),\ell_\varepsilon(t))$ the solutions of the coupled problem with this new external loading, whose existence and uniqueness have been proved above. The reparametrised functions

$(u^\varepsilon(t,x),\ell^\varepsilon(t)) := (u_\varepsilon(\frac{t}{\varepsilon},x),\ell_\varepsilon(\frac{t}{\varepsilon}))$ solve

$$
\begin{aligned}
&\varepsilon^2 u^\varepsilon_{tt}(t,x) - u^\varepsilon_{xx}(t,x) = 0, && t>0,\ 0<x<\ell^\varepsilon(t), && (3.1a)\\
&u^\varepsilon(t,0) = w^\varepsilon(t), && t>0, && (3.1b)\\
&u^\varepsilon(t,\ell^\varepsilon(t)) = 0, && t>0 && (3.1c)\\
&u^\varepsilon(0,x) = u^\varepsilon_0(x), && 0<x<\ell_0, && (3.1d)\\
&u^\varepsilon_t(0,x) = u^\varepsilon_1(x), && 0<x<\ell_0. && (3.1e)
\end{aligned}
$$

Notice that the speed of sound is now $\frac{1}{\varepsilon}$ and that the data may depend on ε. We require that

$$
w^\varepsilon \in \widetilde{C}^{0,1}(0,+\infty), \quad u^\varepsilon_0 \in C^{0,1}([0,\ell_0]), \quad u^\varepsilon_1 \in L^\infty(0,\ell_0), \tag{3.2a}
$$

and impose the compatibility conditions

$$
u^\varepsilon_0(0) = w^\varepsilon(0), \quad u^\varepsilon_0(\ell_0) = 0. \tag{3.2b}
$$

We also assume that

$$
\begin{aligned}
&w^\varepsilon \overset{*}{\rightharpoonup} w \text{ weakly* in } W^{1,\infty}(0,T), && (3.3a)\\
&u^\varepsilon_0 \text{ is bounded in } W^{1,\infty}(0,\ell_0), && (3.3b)\\
&\varepsilon u^\varepsilon_1 \text{ is bounded in } L^\infty(0,\ell_0). && (3.3c)
\end{aligned}
$$

As in the previous section, one then introduces the internal energy

$$
\begin{aligned}
\mathcal{E}^\varepsilon(t;\ell^\varepsilon,w^\varepsilon) =& \frac{1}{2}\int_0^{\ell(t)} u^\varepsilon_t(t,x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u^\varepsilon_x(t,x)^2\,\mathrm{d}t\\
=& \frac{1}{\varepsilon}\int_t^{t+\varepsilon\ell^\varepsilon(t)} [\varepsilon\dot w^\varepsilon(s) - \dot f^\varepsilon(s)]^2\,\mathrm{d}s + \frac{1}{\varepsilon}\int_{t-\varepsilon\ell^\varepsilon(t)}^t \dot f^\varepsilon(s)^2\,\mathrm{d}s,
\end{aligned}
$$

where f^ε is related to u^ε as in Proposition 1.2. We will use the following generalisation of (1.5):

$$
f^\varepsilon(s) = \varepsilon w^\varepsilon(s) - \frac{\varepsilon}{2}u^\varepsilon_0(\tfrac{s}{\varepsilon}) - \frac{\varepsilon^2}{2}\int_0^{\frac{s}{\varepsilon}} u^\varepsilon_1(x)\,\mathrm{d}x - \varepsilon w^\varepsilon(0) + \frac{\varepsilon}{2}u^\varepsilon_0(0), \text{ for every } s\in[0,\varepsilon\ell_0], \tag{3.4a}
$$

$$
f^\varepsilon(s) = \frac{\varepsilon}{2}u^\varepsilon_0(-\tfrac{s}{\varepsilon}) - \frac{\varepsilon^2}{2}\int_0^{-\frac{s}{\varepsilon}} u^\varepsilon_1(x)\,\mathrm{d}x - \frac{\varepsilon}{2}u^\varepsilon_0(0), \qquad \text{for every } s\in(-\varepsilon\ell_0,0]. \tag{3.4b}
$$

The dynamic energy release rate is defined as in the previous section and the following formula holds, cf. (1.9):

$$
G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) = 2\frac{1-\varepsilon\dot\ell^\varepsilon(t)}{1+\varepsilon\dot\ell^\varepsilon(t)}\dot f^\varepsilon(t+\varepsilon\ell^\varepsilon(t))^2.
$$

We now notice that the reparametrisation used to obtain (3.1) also affects the local toughness. Indeed, we have

$$
\kappa(\ell_\varepsilon(t),\dot\ell_\varepsilon(t)) = \kappa(\ell^\varepsilon(t),\varepsilon\dot\ell^\varepsilon(t)),
$$

so that Griffith's criterion now reads as follows:

$$
\begin{aligned}
&\dot\ell^\varepsilon(t) \ge 0, && (3.5a)\\
&G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) \le \kappa(\ell^\varepsilon(t),\varepsilon\dot\ell^\varepsilon(t)), && (3.5b)\\
&\Big[G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t),\varepsilon\dot\ell^\varepsilon(t))\Big]\,\dot\ell^\varepsilon(t) = 0. && (3.5c)
\end{aligned}
$$

As above we employ its equivalent form

$$\begin{cases} \dot\ell^\varepsilon(t) = \dfrac{1}{\varepsilon}\dfrac{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 - \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t))}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 + \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t))} \vee 0, & \text{for a.e. } t>0, \\ \ell^\varepsilon(0) = \ell_0. \end{cases} \tag{3.6}$$

In the quasistatic model, we consider the potential energy

$$\mathcal{E}_{qs}(t;\ell,w) := \min\left\{ \frac{1}{2}\int_0^{\lambda(t)} \dot v(x)^2 \,\mathrm{d}x : v\in H^1(0,L),\ v(0)=w(t),\ v(\lambda(t))=0 \right\},$$

where $\dot v$ denotes the derivative of v with respect to x. The quasistatic energy release rate is

$$G_{qs}(t) := -\partial_\ell \mathcal{E}_{qs}(t;\ell,w).$$

The following result was proved in [19, Proposition 2.3, Theorem 2.5, Theorem 2.11] assuming that the toughness only depends on the position in the reference configuration. The extension to the case of speed-dependent toughness requires only minor modifications in the proof. In Remark 3.2 we highlight the only step where the toughness plays a role.

Theorem 3.1. *Let $T>0$. Assume that the toughness κ satisfies* (1.8) *and is upper semicontinuous. Assume that the initial data satisfy* (3.2) *and* (3.3). *Let $(u^\varepsilon, \ell^\varepsilon)$ be the solution to the coupled problem* (3.1) *&* (3.6). *Then, there exists $L>0$ with $\ell^\varepsilon(T)\le L$, a non-decreasing evolution $\ell\colon[0,T]\to[0,L]$, and a sequence ε_k such that*

$$\ell^{\varepsilon_k}(t) \to \ell(t)$$

for every $t\in[0,T]$. Moreover,

$$u^{\varepsilon_k} \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(0,L)),$$

where

$$u(t,x) = \begin{cases} -\frac{w(t)}{\ell(t)}x + w(t), & \text{for a.e. } (t,x)\colon x<\ell(t), \\ 0, & \text{for a.e. } (t,x)\colon x\ge \ell(t). \end{cases}$$

Finally,

$$G_{qs}(t) \le \kappa(\ell(t),0) \tag{3.7}$$

and the quasistatic energy release rate is given by $G_{qs}(t) = \frac{w(t)^2}{2\ell(t)^2}$.

Remark 3.2. We highlight that in the quasistatic limit the toughness appearing in Griffith's criterion is evaluated at debonding speed zero, i.e., it corresponds to the so-called steady state toughness $\kappa(\ell,0)$. Indeed, following the proof of [19, Theorem 2.11] we see that

$$\int_a^b \sqrt{G_{qs}(t)}\,\mathrm{d}t \le \limsup_k \int_a^b \sqrt{G^{\varepsilon_k}(t)}\,\mathrm{d}t \le \limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k\dot\ell^{\varepsilon_k}(t)}\,\mathrm{d}t.$$

for every interval $(a,b)\subset[0,T]$, where the second inequality follows by (3.5b). By the Fatou lemma and the upper semicontinuity of κ, we find

$$\limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k\dot\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \le \int_a^b \limsup_k \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k\dot\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \le \int_a^b \sqrt{\kappa(\ell(t),0)}\,\mathrm{d}t,$$

which yields (3.7)

In this work we observe a particular behaviour of the quasistatic limit by providing two examples. The first example shows that (3.5c) does not pass to the limit as $\varepsilon\to0$, i.e.,

$$[G_{qs}(t) - \kappa(\lambda(t),0)]\,\dot\lambda(t) = 0 \tag{3.8}$$

does *not* hold in general. The second example shows that brutal propagation is possible in the quasistatic limit even if the dynamic toughness penalises high-speed debonding.

We will employ the bounce formula (1.4) in the following form:

$$f^\varepsilon(s) = \varepsilon w^\varepsilon(s) + f^\varepsilon(\omega_\varepsilon(s)), \tag{3.9}$$

where

$$\omega_\varepsilon := \varphi_\varepsilon \circ \psi_\varepsilon^{-1}, \quad \varphi_\varepsilon(s) := s - \varepsilon\ell^\varepsilon(s), \quad \psi_\varepsilon(s) := s + \varepsilon\ell^\varepsilon(s).$$

Notice that

$$\dot\omega_\varepsilon^{-1}(s) = \frac{1 + \varepsilon\dot\ell^\varepsilon(\psi_\varepsilon^{-1}(s))}{1 - \varepsilon\dot\ell^\varepsilon(\psi_\varepsilon^{-1}(s))}. \tag{3.10}$$

(See Figure 2.)

3.1. **Example 1: the activation condition fails.** We now show that the presence of a regularising term in Griffith's criterion, given by an explicit dependence of the local toughness upon the debonding speed, is not in general sufficient to guarantee the convergence of (3.5c).

We consider here a local toughness given by

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t)) := \frac{1}{2} + c_3\varepsilon\dot\ell^\varepsilon(t), \tag{3.11}$$

with $c_3 > 0$. Griffith's activation condition reads as

$$\left[G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) - \frac{1}{2} - c_3\varepsilon\dot\ell^\varepsilon(t)\right]\dot\ell^\varepsilon(t) = 0,$$

which has the same form of the corresponding equation in inertia-free fracture models with a viscous regularisation [16, 22].

Notice that the choice $\kappa = \tilde\kappa := \frac{1}{2}$ was precisely the one employed in [19, Section 3] and we will henceforth refer to $(v^\varepsilon, \lambda^\varepsilon)$ as the dynamic solutions analysed in that paper and to (v, λ) as their limit as $\varepsilon \to 0$.

Using (3.11), we write (3.6) in normal form, obtaining

$$\begin{cases} \dot\ell^\varepsilon(t) = \frac{1}{\varepsilon}\frac{4\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2-1}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2+\frac{1}{2}+c_3+\sqrt{(2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2+\frac{1}{2}-c_3)^2+16c_3\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2}} \vee 0, & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{3.12}$$

We set $\ell_0 := 2$, $w^\varepsilon(t) := t$, $u_1^\varepsilon := 1$, and

$$u_0^\varepsilon(x) := \begin{cases} (2\varepsilon\lfloor\frac{1}{\varepsilon}\rfloor - \sqrt{1+\varepsilon^2})x + 2(\sqrt{1+\varepsilon^2} - \varepsilon\lfloor\frac{1}{\varepsilon}\rfloor), & 0 \le x \le 1, \\ -\sqrt{1+\varepsilon^2}\,x + 2\sqrt{1+\varepsilon^2}, & 1 \le x \le 2. \end{cases}$$

The very same data were chosen in [19, Section 3]. This choice of u_0^ε gives rise to two different alternating behaviours in the propagation of the debonding front, since the derivative of f^ε takes two values, cf. (3.4):

$$\dot f^\varepsilon(t) = \begin{cases} \dot f_1^\varepsilon := \dfrac{\varepsilon + \sqrt{1+\varepsilon^2}}{2}, & -2\varepsilon \le t \le -\varepsilon, \\ \dot f_2^\varepsilon := \dfrac{\varepsilon + \sqrt{1+\varepsilon^2}}{2} - \varepsilon\lfloor\dfrac{1}{\varepsilon}\rfloor, & -\varepsilon \le t \le \varepsilon, \\ \dot f_3^\varepsilon := \dot f_1^\varepsilon, & \varepsilon \le t \le 2\varepsilon. \end{cases}$$

For every $i \ge 1$ we call ℓ_i^ε the solution of (3.12) when $\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t)) = \dot f_i^\varepsilon$.

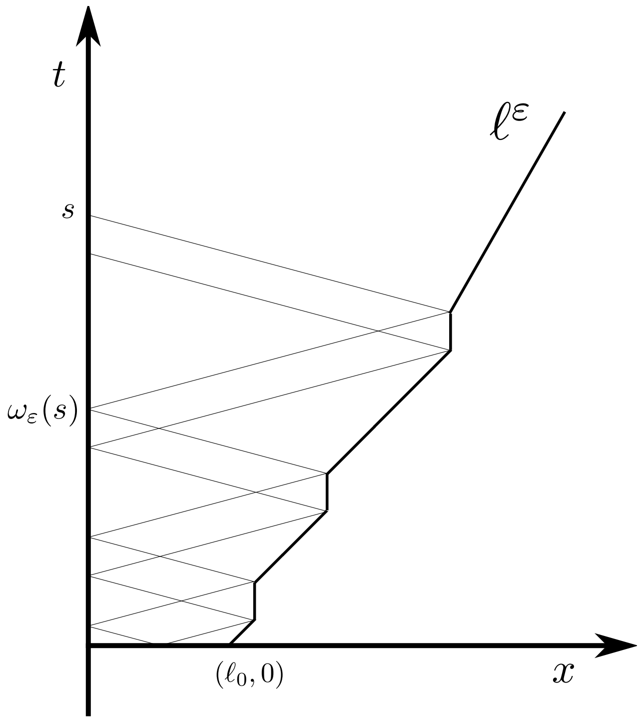

FIGURE 2. The evolution of ℓ^ε in Example 1 is represented by a zig-zag with alternation between phases of propagation of the debonding front and stop phases.

We notice that, by plugging $\dot f_2^\varepsilon$ in (3.12), we have $\dot\ell_2^\varepsilon = 0$. As a consequence of (1.4), it results that $\dot f^\varepsilon$ and $\dot\ell^\varepsilon$ are piecewise constant in $[0, +\infty)$; we denote by $\dot f_i^\varepsilon$, $\dot\ell_i^\varepsilon$ their values, indexed increasingly with respect to time, see Figure 2. The rule for the update of $\dot f^\varepsilon$ reads

$$\dot f_{i+3}^\varepsilon = \varepsilon + \frac{1 - \varepsilon\dot\ell_i^\varepsilon}{1 + \varepsilon\dot\ell_i^\varepsilon}\dot f_i^\varepsilon, \quad \text{for } i \geq 1.$$

Hence, $\dot f_5^\varepsilon = \dot f_2^\varepsilon + \varepsilon$. By direct computation it is possible to prove that $\dot\ell_5^\varepsilon = 0$ and that for every $0 \leq i < \lfloor \frac{1}{\varepsilon} \rfloor =: n^\varepsilon$ we have $\dot f_{3i+2}^\varepsilon = \dot f_2^\varepsilon + i\varepsilon$ and $\dot\ell_{3i+2}^\varepsilon = 0$. Thus, the indices $3i + 2$ correspond to stop phases with no propagation of the debonding front until a certain threshold is reached.

In contrast, we have propagation phases for the indices $3i + 1$ and $3i + 3$. Indeed, starting from (3.12), we deduce that

$$\frac{1}{\varepsilon}\frac{2\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 - \tilde\kappa}{2\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 + \tilde\kappa + 2\sqrt{c_3}\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))} \leq \dot\ell^\varepsilon(t) \leq \frac{1}{\varepsilon}\frac{2\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 - \tilde\kappa}{2\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 + \tilde\kappa}. \tag{3.13}$$

In the previous chain of inequalities, the first is obtained from (3.12) by using the fact that $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$; the second is obtained by ignoring the term $16c_3\dot f^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2$ in the denominator of (3.12). Therefore, from the first inequality of (3.13) we obtain

$$\dot\ell_3^\varepsilon(t) = \dot\ell_1^\varepsilon(t) \geq \frac{1}{\varepsilon}\frac{2(\dot f_1^\varepsilon)^2 - \tilde\kappa}{2(\dot f_1^\varepsilon)^2 + \tilde\kappa + 2\sqrt{c_3}\dot f_1^\varepsilon} = \frac{\varepsilon + \sqrt{1 + \varepsilon^2}}{\varepsilon^2 + \varepsilon\sqrt{1 + \varepsilon^2} + 1 + \sqrt{c_3}(\varepsilon + \sqrt{1 + \varepsilon^2})} \geq \frac{1}{2 + \sqrt{c_3}},$$

where the last inequality holds for ε sufficiently small. On the other hand, the second inequality of (3.13) implies that the debonding speed is controlled by the corresponding to constant toughness $\tilde\kappa$:

$$\dot\ell^\varepsilon(t) \le \dot\lambda^\varepsilon(t), \quad \text{for a.e. } t>0.$$

Since the function $x \mapsto \frac{1-\varepsilon x}{1+\varepsilon x}$ is non-increasing, then

$$\dot f_6^\varepsilon = \dot f_4^\varepsilon = \varepsilon + \frac{1-\varepsilon\dot\ell_1^\varepsilon}{1+\varepsilon\dot\ell_1^\varepsilon}\dot f_1^\varepsilon \ge \varepsilon + \frac{1-\varepsilon\dot\lambda_1^\varepsilon}{1+\varepsilon\dot\lambda_1^\varepsilon}\dot f_1^\varepsilon = \dot f_1^\varepsilon,$$

where the last equality follows from the explicit expression of the debonding speed for toughness $\tilde\kappa$ in the first interval, $\dot\lambda_1^\varepsilon = 1/\sqrt{1+\varepsilon^2}$, obtained by plugging $\kappa=\tilde\kappa$ into (3.6). We iterate this argument and obtain for every $i\ge 1$

$$\dot f_{3i+3}^\varepsilon = \dot f_{3i+1}^\varepsilon = \varepsilon + \frac{1-\varepsilon\dot\ell_{3i-2}^\varepsilon}{1+\varepsilon\dot\ell_{3i-2}^\varepsilon}\dot f_{3i-2}^\varepsilon \ge \varepsilon + \frac{1-\varepsilon\dot\lambda_{3i-2}^\varepsilon}{1+\varepsilon\dot\lambda_{3i-2}^\varepsilon}\dot f_1^\varepsilon = \dot f_1^\varepsilon. \tag{3.14}$$

Moreover, by (3.13) we recall that $\dot\ell^\varepsilon(t) \ge \frac{1}{\varepsilon}H(\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t)))$, where $H(x) := \frac{2x^2-\tilde\kappa}{2x^2+\tilde\kappa+2\sqrt{c_3}x}$. We have

$$H'(x) = \frac{1}{\varepsilon}\frac{4\sqrt{c_3}x^2+4x+\sqrt{c_3}}{(2x^2+\frac12+2\sqrt{c_3}x)^2} \ge 0, \quad \text{for } x\ge 0.$$

(Recall that $\tilde\kappa=\frac12$ and notice that the second order polynomial at its numerator has negative roots.) Therefore, by (3.14), we get

$$\dot\ell_{3i+3}^\varepsilon = \dot\ell_{3i+3}^\varepsilon \ge H(\dot f_{3i+1}^\varepsilon) \ge H(\dot f_1^\varepsilon) \ge \frac{1}{2+\sqrt{c_3}} =: \nu,$$

for ε small enough.

Summarising, in the first $3n^\varepsilon$ iterations we observe the alternation of two phases:

- stop phases, where the debonding speed is zero,
- propagation phases, where the debonding speed is uniformly bounded from below.

So far, we have not insisted on detailing the time intervals where $\dot\ell^\varepsilon$ is zero or positive. Let us just notice that the length of those intervals is determined by the rule for the update of f^ε, see also (3.9). By (3.10), we obtain that in the iterative scheme outlined above the length of the time intervals is dilated by a factor $\frac{1+\varepsilon\dot\ell_i^\varepsilon}{1-\varepsilon\dot\ell_i^\varepsilon}$, see Figure 3. This shows that the intervals where $\dot\ell^\varepsilon=0$ have all the same length $2\varepsilon=\varepsilon\ell_0$. In contrast, the length of the intervals where $\dot\ell^\varepsilon\ne 0$ is increasing, since at the i-th iteration those intervals are dilated by a factor $\frac{1+\varepsilon\dot\ell_{3i+1}^\varepsilon}{1-\varepsilon\dot\ell_{3i+1}^\varepsilon} = \frac{1+\varepsilon\dot\ell_{3i+3}^\varepsilon}{1-\varepsilon\dot\ell_{3i+3}^\varepsilon} \ge \frac{1+\varepsilon\nu}{1-\varepsilon\nu}$.

Following a similar iterative scheme, we now construct a fictitious zig-zag evolution $\gamma^\varepsilon(t)$ such that $\gamma^\varepsilon(0)=\ell_0=2$ and $\dot\gamma^\varepsilon\in\{0,\nu\}$. More precisely, imitating the construction of ℓ^ε, we set

$$\dot\gamma^\varepsilon(t) = \begin{cases} \nu, & \text{if } t-\gamma^\varepsilon(t)\in(-2\varepsilon,-\varepsilon),\\ 0, & \text{if } t-\gamma^\varepsilon(t)\in(-\varepsilon,\varepsilon),\\ \nu, & \text{if } t-\gamma^\varepsilon(t)\in(\varepsilon,2\varepsilon).\end{cases}$$

This defines γ^ε in $[0,s_1^\varepsilon]$, where s_1^ε denotes the time such that $s_1^\varepsilon-\gamma^\varepsilon(s_1^\varepsilon)=2\varepsilon$. It turns out that $s_1^\varepsilon = 2\varepsilon(2-\varepsilon\nu)/(1-\varepsilon\nu)$, see Figure 3. Next we repeat this pattern with the following rule: at each iteration the intervals where $\dot\ell^\varepsilon=0$ maintain the same length 2ε; the two intervals where $\dot\ell^\varepsilon\ne0$ are dilated by the fixed factor $\frac{1+\varepsilon\nu}{1-\varepsilon\nu}$. By construction we obtain

$$\gamma^\varepsilon(t) \le \ell^\varepsilon(t) \le \lambda^\varepsilon(t), \tag{3.15}$$

where the latter inequality follows by (3.13). More precisely, let us denote by s_i^ε the extremum of the interval where γ^ε is defined after the $(i-1)$-th iteration, obtained replicating s_1^ε. For every

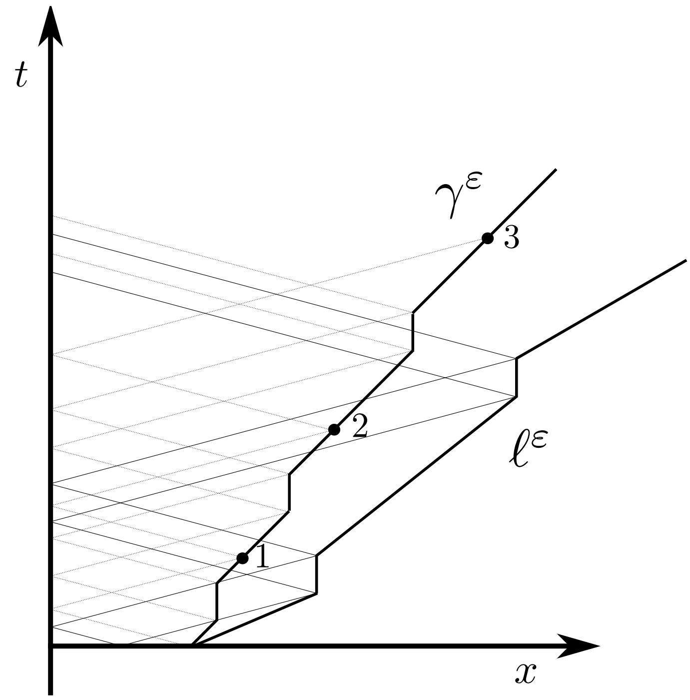

FIGURE 3. Evolution of γ^ε and ℓ^ε in Example 1. The points $i = 1, 2, 3$ in figure represent the points $(s_i^\varepsilon, \gamma^\varepsilon(s_i^\varepsilon))$ in the construction of γ^ε.

$i = 1, \dots, n^\varepsilon$,

$$s_i^\varepsilon = 2\varepsilon i + \frac{2\varepsilon}{1-\varepsilon\nu}\sum_{j=0}^{i-1} d_\varepsilon^j = 2\varepsilon i + \frac{2\varepsilon}{1-\varepsilon\nu}\frac{1-d_\varepsilon^i}{1-d_\varepsilon}, \tag{3.16}$$

where

$$d_\varepsilon := \frac{1+\varepsilon\nu}{1-\varepsilon\nu}.$$

The first summand in (3.16) corresponds to the total length of all intervals where $\dot\gamma^\varepsilon = 0$ up to s_i^ε, while the second accounts for the intervals where $\dot\gamma^\varepsilon = \nu$. The position of the debonding front at time s_i^ε is

$$\gamma^\varepsilon(s_i^\varepsilon) = 2 + \frac{2\varepsilon\nu}{1-\varepsilon\nu}\frac{1-d_\varepsilon^i}{1-d_\varepsilon}.$$

We consider the map $i \mapsto x = \gamma^\varepsilon(s_i^\varepsilon)$ for $i = 1, \dots, n^\varepsilon$ and its inverse

$$i^\varepsilon(x) = \left\lfloor \frac{\log(x-1)}{\log d_\varepsilon} \right\rfloor.$$

In order to understand the limit behaviour of ℓ^ε, we study the limit of γ^ε. (Notice that their pointwise limits are both uniform limits.) A straightforward computation shows that

$$\varepsilon i^\varepsilon(x) \quad \stackrel{\varepsilon\to 0}{\longrightarrow} \quad \frac{1}{2\nu}\log(x-1) = \frac{2+\sqrt{c_3}}{2}\log(x-1).$$

Moreover,

$$d_\varepsilon^{i^\varepsilon(x)} \stackrel{\varepsilon\to 0}{\longrightarrow} x-1.$$

We now let $\varepsilon \to 0$ in the expression for $s^\varepsilon_{i^\varepsilon(x)}$ and find the expression for the inverse $t \mapsto \gamma(t)$:

$$\gamma^{-1}(x) = \lim_{\varepsilon\to 0} s^\varepsilon_{i^\varepsilon(x)} = (2+\sqrt{c_3})\,[\log(x-1)+x-2]\,. \tag{3.17}$$

Notice also that

$$s^\varepsilon_{n^\varepsilon} \xrightarrow{\varepsilon\to 0} 2 + \frac{e^{2\nu}-1}{\nu}, \quad \gamma^\varepsilon(s^\varepsilon_{n^\varepsilon}) \xrightarrow{\varepsilon\to 0} 1+e^{2\nu}.$$

Finally, passing to the limit in (3.15), we obtain

$$\lambda^{-1}(x) \le \ell^{-1}(x) \le \gamma^{-1}(x), \quad \text{for } x \le 1+e^{2\nu}. \tag{3.18}$$

The explicit expression for λ^{-1} derived in [19, Section 3] gives

$$\lambda^{-1}(x) = \frac{1}{2+\sqrt{c_3}}\,\gamma^{-1}(x). \tag{3.19}$$

This shows that $t \mapsto \ell(t)$ cannot satisfy Griffith's quasistatic criterion. Indeed, by (3.13), (3.17), and (3.19), the debonding speed is uniformly bounded by one, so ℓ has no jumps. Moreover, since a Griffith evolution must satisfy (3.8), we would have

$$\frac{t^2}{2\ell(t)^2} = \kappa(\ell,0) = \frac{1}{2}, \quad \text{if } \dot\ell > 0,$$

whence $\ell(t) = t$. This is incompatible with (3.18) and therefore the limit evolution $t \mapsto \ell$ does not satisfy Griffith's activation condition (3.8).

Notice also that the same behaviour may be observed even with a toughness such that

$$\lim_{\mu\to 1^-} \kappa(x,\mu) = +\infty, \quad \text{for every } x. \tag{3.20}$$

Indeed, in the previous example $\dot\ell^\varepsilon$ is uniformly bounded by one, thus $\mu = \varepsilon\dot\ell^\varepsilon \le \varepsilon$. If we consider a toughness satisfying (3.20) and such that it coincides with (3.11) for $\mu = \varepsilon\dot\ell^\varepsilon \le c < 1$, we obtain the same counterexample.

3.2. **Example 2: brutal propagation.** When the local toughness κ satisfies (3.20), high-speed propagations are penalised at the dynamic level (for $\varepsilon > 0$). Therefore, one may ask if such a property prevents brutal propagations, i.e., time discontinuities, in the quasistatic limit as $\varepsilon \to 0$. We now prove that the answer is negative, more precisely we show a case where (3.20) holds and the limit evolution jumps in time.

Let us consider a local toughness of the form

$$\kappa(x,\mu) := \tilde\kappa(x)\frac{1+\mu}{1-\mu}, \tag{3.21}$$

where $\tilde\kappa$ is independent of μ. This example is reported in [13, Section 7.4]. Here we assume that

$$\tilde\kappa(x) := \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \bar x, \\ \frac{1}{8}, & \text{if } x > \bar x, \end{cases}$$

where $\bar x > \ell_0 := 2$. Notice that $\tilde\kappa$ is non-increasing and takes only two values, modelling a composite material. The role of toughness discontinuities was analysed in some examples in [10, 20], in the case where the toughness depends only on the position x; in the quasistatic limit, those examples display brutal propagation as soon as the debonding front meets the toughness discontinuity. We show that this behaviour is not ruled out by (3.20).

For $0 < \varepsilon \ll 1$ we take affine initial data:

$$u^\varepsilon_0(x) := -(\varepsilon+1)x + 2(\varepsilon+1), \quad u^\varepsilon_1(x) := 1.$$

We set

$$w^\varepsilon(t) := t + 2(\varepsilon + 1).$$

By (3.4) we find that

$$\dot f^\varepsilon(t) = \dot f_1^\varepsilon := \varepsilon + \frac{1}{2}, \quad \text{for a.e. } t \in (-2\varepsilon, 2\varepsilon).$$

We can then solve (3.6) as long as $t - \varepsilon\ell^\varepsilon(t) \in (-2\varepsilon, 2\varepsilon)$: it turns out that $\dot\ell^\varepsilon(t) = \dot\ell_1^\varepsilon$, where $\dot\ell_1^\varepsilon$ is independent of time and satisfies, by (3.21),

$$\dot\ell_1^\varepsilon = \frac{1}{\varepsilon}\,\frac{2(\dot f_1^\varepsilon)^2 - \dfrac{1}{2}\,\dfrac{1+\varepsilon\dot\ell_1^\varepsilon}{1-\varepsilon\dot\ell_1^\varepsilon}}{2(\dot f_1^\varepsilon)^2 + \dfrac{1}{2}\,\dfrac{1+\varepsilon\dot\ell_1^\varepsilon}{1-\varepsilon\dot\ell_1^\varepsilon}} \vee 0.$$

By explicitly solving the last equation as a second order polynomial equation in $\dot\ell_1^\varepsilon$ and recalling that any solution must fulfil $\varepsilon\dot\ell^\varepsilon < 1$, we find

$$\dot\ell_1^\varepsilon = \frac{1}{\varepsilon}\,\frac{2\dot f_1^\varepsilon - 1}{2\dot f_1^\varepsilon + 1} = \frac{1}{\varepsilon + 1}.$$

We have thus found the evolution of $t \mapsto \ell^\varepsilon(t)$ for a.e. $t \in (0, s_1^\varepsilon)$, where s_1^ε is defined by $\varphi_\varepsilon(s_1^\varepsilon) = s_1^\varepsilon - \ell^\varepsilon(s_1^\varepsilon) = 2\varepsilon =: t_1^\varepsilon$. We extend f^ε using (3.9): for a.e. $t \in (t_1^\varepsilon, t_2^\varepsilon)$, where $t_2^\varepsilon = \omega_\varepsilon^{-1}(t_1^\varepsilon) = \varphi_\varepsilon(\psi_\varepsilon^{-1}(t_1^\varepsilon))$, we have

$$\dot f^\varepsilon(t) = \dot f_2^\varepsilon := \varepsilon + \frac{1-\varepsilon\dot\ell_1^\varepsilon}{1+\varepsilon\dot\ell_1^\varepsilon}\dot f_1^\varepsilon = \varepsilon + \frac{1 - \frac{2\dot f_1^\varepsilon - 1}{2\dot f_1^\varepsilon + 1}}{1 + \frac{2\dot f_1^\varepsilon - 1}{2\dot f_1^\varepsilon + 1}}\dot f_1^\varepsilon = \varepsilon + \frac{1}{2} = \dot f_1^\varepsilon.$$

Again, one solves (3.6) to find $\dot\ell^\varepsilon$ in $(s_1^\varepsilon, s_2^\varepsilon)$, where $\varphi_\varepsilon(s_2^\varepsilon) = t_2^\varepsilon$. We obtain

$$\dot\ell^\varepsilon(t) = \dot\ell_2^\varepsilon := \frac{1}{\varepsilon}\,\frac{2\dot f_2^\varepsilon - 1}{2\dot f_2^\varepsilon + 1} = \frac{1}{\varepsilon+1} = \dot\ell_1^\varepsilon.$$

We may iterate this argument up to the time $\bar t^\varepsilon$ such that $\ell^\varepsilon(\bar t^\varepsilon) = \bar x$. Thus we find $\dot f^\varepsilon(t) = \dot f_1^\varepsilon$ for $t \in (-2\varepsilon, \bar t^\varepsilon + \varepsilon\bar x)$ and $\dot\ell^\varepsilon(t) = \dot\ell_1^\varepsilon$ for $t \in (0, \bar t^\varepsilon)$.

When the evolution of the debonding front reaches the toughness discontinuity $\bar x$, we observe an abrupt change. According to (3.21), the equation for ℓ^ε is now

$$\dot\ell^\varepsilon(t) = \frac{1}{\varepsilon}\,\frac{2(\dot f_1^\varepsilon)^2 - \dfrac{1}{8}\,\dfrac{1+\varepsilon\dot\ell^\varepsilon(t)}{1-\varepsilon\dot\ell^\varepsilon(t)}}{2(\dot f_1^\varepsilon)^2 + \dfrac{1}{8}\,\dfrac{1+\varepsilon\dot\ell^\varepsilon(t)}{1-\varepsilon\dot\ell^\varepsilon(t)}}, \quad \text{for } t - \varepsilon\ell^\varepsilon(t) \in (\bar t^\varepsilon, \bar t^\varepsilon + \varepsilon\bar x).$$

As before, in that interval $\dot\ell^\varepsilon$ is a constant found by solving the second order polynomial equation in $\dot\ell^\varepsilon$. It turns out that

$$\dot\ell^\varepsilon = \frac{1}{\varepsilon}\,\frac{4\dot f_2^\varepsilon - 1}{4\dot f_2^\varepsilon + 1} = \frac{1}{\varepsilon}\,\frac{4\varepsilon + 1}{4\varepsilon + 3}.$$

Since $\dot\ell^\varepsilon \sim \frac{1}{3\varepsilon}$ as $\varepsilon \to 0$, such fast propagation in the evolution of the debonding front $t \mapsto \ell^\varepsilon(t)$ converges to a jump of finite size in the limit evolution. This proves that (3.20) does not prevent brutal propagation in the quasistatic limit.

It is remarkable that the effective toughness during the fast propagation is given by

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)) = \frac{1}{8}\,\frac{1+\varepsilon\dot{\ell}^\varepsilon(t)}{1-\varepsilon\dot{\ell}^\varepsilon(t)} \sim \frac{1}{8}\,\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{4},$$

while in the previous phase it is

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)) = \frac{1}{2}\,\frac{1+\varepsilon\dot{\ell}^\varepsilon(t)}{1-\varepsilon\dot{\ell}^\varepsilon(t)} \sim \frac{1}{2}.$$

This shows that the brutal propagation is due to a decrease of the effective toughness from $\frac{1}{2}$ to $\frac{1}{4}$. On the other hand, the limit value of the toughness is greater than the value predicted by Griffith's quasistatic criterion, i.e., the steady state toughness $\kappa(\ell, 0) = \frac{1}{8}$.

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References

[1] B. Bourdin, G. A. Francfort, and J.-J. Marigo, *The variational approach to fracture*, J. Elasticity, 91 (2008), pp. 5–148.

[2] A. Chambolle, *A density result in two-dimensional linearized elasticity, and applications*, Arch. Ration. Mech. Anal., 167 (2003), pp. 211–233.

[3] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal., 176 (2005), pp. 165–225.

[4] G. Dal Maso and C. J. Larsen, *Existence for wave equations on domains with arbitrary growing cracks*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 22 (2011), pp. 387–408.

[5] G. Dal Maso, C. J. Larsen, and R. Toader, *Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition*, J. Mech. Phys. Solids, 95 (2016), pp. 697–707.

[6] G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257–290.

[7] G. Dal Maso, G. Lazzaroni, and L. Nardini, *Existence and uniqueness of dynamic evolutions for a peeling test in dimension one*, J. Differential Equations, 261 (2016), pp. 4897–4923.

[8] G. Dal Maso and I. Lucardesi, *The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on the data*, Appl. Math. Res. Express, in press (2016), doi: 10.1093/amrx/abw006.

[9] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Rational Mech. Anal., 162 (2002), pp. 101–135.

[10] P.-E. Dumouchel, J.-J. Marigo, and M. Charlotte, *Dynamic fracture: an example of convergence towards a discontinuous quasistatic solution*, Contin. Mech. Thermodyn., 20 (2008), pp. 1–19.

[11] G. A. Francfort and C. J. Larsen, *Existence and convergence for quasi-static evolution in brittle fracture*, Comm. Pure Appl. Math., 56 (2003), pp. 1465–1500.

[12] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.

[13] L. B. Freund, *Dynamic fracture mechanics*, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Cambridge, 1990.

[14] T. Goldman, A. Livne, and J. Fineberg, *Acquisition of inertia by a moving crack*, Phys. Rev. Lett., 104 (2010), article number 114301.

[15] J. K. Hale, *Ordinary differential equations*, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., second ed., 1980.

[16] D. Knees, A. Mielke, and C. Zanini, *On the inviscid limit of a model for crack propagation*, Math. Models Methods Appl. Sci. 18 (2008), pp. 1529–1569.

[17] C. M. Landis, T. Pardoen, and J. W. Hutchinson, *Crack velocity dependent toughness in rate dependent materials*, Mech. Mater., 32 (2000), pp. 663–678.

[18] C. J. Larsen, C. Ortner, and E. Süli, *Existence of solutions to a regularized model of dynamic fracture*, Math. Models Methods Appl. Sci., 20 (2010), pp. 1021–1048.
[19] G. Lazzaroni and L. Nardini, *On the quasistatic limit of dynamic evolutions for a peeling test in dimension one*, J. Nonlinear Sci., (2017), doi:10.1007/s00332-017-9407-0.
[20] G. Lazzaroni, R. Bargellini, P.-E. Dumouchel, and J.-J. Marigo, *On the role of kinetic energy during unstable propagation in a heterogeneous peeling test*, Int. J. Fract., 175 (2012), pp. 127–150.
[21] G. Lazzaroni, R. Rossi, M. Thomas, and R. Toader, *Rate-independent damage in thermo-viscoelastic materials with inertia*, Preprint SISSA 52/2014/MATE.
[22] G. Lazzaroni and R. Toader, *A model for crack propagation based on viscous approximation*, Math. Models Methods Appl. Sci., 21 (2011), pp. 2019–2047.
[23] Y. Lee and V. Prakash, *Dynamic fracture toughness versus crack tip speed relationship at lower than room temperature for high strength 4340VAR structural steel*, J. Mech. Phys. Solids, 46 (1998), pp. 1943–1967.
[24] M. Marder, *New dynamical equation for cracks*, Phys. Rev. Lett., 66 (1991), pp. 2484–2487.
[25] D. P. Miannay, *Time-dependent fracture mechanics*, Mechanical Engineering Series, Springer-Verlag, New York, 2001.
[26] A. Mielke and T. Roubíček, *Rate-Independent Systems: Theory and Application*, Applied Mathematical Sciences, 193, Springer, New York (2015).
[27] L. Nardini, *A note on the convergence of singularly perturbed second order potential-type equations*, J. Dyn. Differ. Equations, 29 (2017), pp. 783–797.
[28] S. Nicaise and A.-M. Sändig, *Dynamic crack propagation in a 2D elastic body: the out-of-plane case*, J. Math. Anal. Appl., 329 (2007), pp. 1–30.
[29] R. Rossi and T. Roubíček, *Thermodynamics and analysis of rate-independent adhesive contact at small strains*, Nonlinear Anal., 74 (2011), pp. 3159–3190.
[30] R. Rossi and M. Thomas, *From adhesive to brittle delamination in visco-elastodynamics*, WIAS Preprint 2259 (2016).
[31] T. Roubíček, *Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity*, SIAM J. Math. Anal., 45 (2013), pp. 101–126.
[32] T. W. Webb and E. C. Aifantis, *Oscillatory fracture in polymeric materials*, Int. J. Solids Structures, 32 (1995), pp. 2725–2743.

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