

ON FRACTIONAL HARDY INEQUALITIES IN CONVEX SETS

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ABSTRACT. We prove a Hardy inequality on convex sets, for fractional Sobolev-Slobodeckii spaces of order (s, p) . The proof is based on the fact that in a convex set the distance from the boundary is a superharmonic function, in a suitable sense. The result holds for every $1 < p < \infty$ and $0 < s < 1$, with a constant which is stable as s goes to 1.

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1. INTRODUCTION

1.1. A quick overview on Hardy inequality. Given an open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, we will use the notation

$$d_\Omega(x) := \begin{cases} \inf_{y \in \partial\Omega} |x - y|, & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

A fundamental result in the theory of Sobolev spaces is the *Hardy inequality*

$$(1.1) \quad C_\Omega \int_\Omega \frac{|u|^2}{d_\Omega^2} dx \leq \int_\Omega |\nabla u|^2 dx, \quad \text{for every } u \in C_0^\infty(\Omega),$$

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see for example [16, Theorem 21.3]. It is well-known that for a convex set $K \subset \mathbb{R}^N$, such an inequality holds with the dimension-free universal constant

$$C_K = \frac{1}{4},$$

see for example [6, Theorem 2]. Moreover, such a constant is sharp. In order to explain the aims and techniques of the present paper, it is useful to recall a proof of this fact.

A very elegant way of proving (1.1) with sharp constant for convex sets, consists in mimicking *Moser's logarithmic estimate* for positive supersolutions of elliptic partial differential equations. The starting point is the observation that on a convex set K we have $-\Delta d_K \geq 0$, i.e. the distance function is superharmonic. More precisely, it holds

$$(1.2) \quad \int_K \langle \nabla d_K, \nabla \varphi \rangle dx \geq 0, \quad \text{for every nonnegative } \varphi \in C_0^\infty(K).$$

By following Moser (see [15, page 586]), we can test the equation (1.2) with

$$\varphi = \frac{u^2}{d_K},$$

where $u \in C_0^\infty(K)$. This gives

$$2 \int_K \left\langle \frac{\nabla d_K}{d_K}, \nabla u \right\rangle u dx - \int_K \left| \frac{\nabla d_K}{d_K} \right|^2 u^2 dx \geq 0,$$

that is

$$\int_K \left| \frac{\nabla d_K}{d_K} \right|^2 u^2 dx \leq 2 \int_K \left\langle \frac{\nabla d_K}{d_K}, \nabla u \right\rangle u dx.$$

We now use Young's inequality

$$\langle a, b \rangle \leq \frac{|a|^2}{2} + \frac{|b|^2}{2}, \quad a, b \in \mathbb{R}^N,$$

with the following choices

$$a = \sqrt{\delta} \frac{\nabla d_K}{d_K} u \quad \text{and} \quad b = \frac{1}{\sqrt{\delta}} \nabla u,$$

where δ is an arbitrary positive real number. This leads to

$$\int_K \left| \frac{\nabla d_K}{d_K} \right|^2 u^2 dx \leq \delta \int_K \left| \frac{\nabla d_K}{d_K} \right|^2 u^2 dx + \frac{1}{\delta} \int_K |\nabla u|^2 dx,$$

which can be recast as

$$\delta(1 - \delta) \int_K \left| \frac{\nabla d_K}{d_K} \right|^2 u^2 dx \leq \int_K |\nabla u|^2 dx.$$

It is now sufficient to observe that the term $\delta(1 - \delta)$ in the left-hand side is maximal for $\delta = 1/2$. This leads to the Hardy inequality with the claimed sharp constant $1/4$, once it is observed that

$$|\nabla d_K| = 1, \quad \text{a. e. in } K.$$

The latter implies that more generally for every $1 < p < \infty$ we have $-\Delta_p d_K \geq 0$ in K , i.e. d_K is p -superharmonic in the following sense

$$\int_K \langle |\nabla d_K|^{p-2} \nabla d_K, \nabla \varphi \rangle dx \geq 0, \quad \text{for every nonnegative } \varphi \in C_0^\infty(K).$$

By testing this with $\varphi = |u|^p/d_K^{p-1}$ and suitably adapting the proof above, one can prove the more general Hardy inequality for convex sets

$$(1.3) \quad \left(\frac{p-1}{p}\right)^p \int_K \frac{|u|^p}{d_K^p} dx \leq \int_K |\nabla u|^p dx.$$

Once again, the constant appearing in (1.3) is sharp and independent of both K and the dimension N .

1.2. Main result. The scope of the present paper is to prove a *fractional version* of Hardy inequality for convex sets, by adapting to the fractional setting the Moser-type proof presented above. An essential feature of our method is that the relevant constant appearing in the Hardy inequality *is stable as the fractional order of differentiability s converges to 1*, see Remark 1.3 below. More precisely, we prove the following

Theorem 1.1 (Hardy inequality on convex sets). *Let $1 < p < \infty$ and $0 < s < 1$. Let $K \subset \mathbb{R}^N$ be an open convex set such that $K \neq \mathbb{R}^N$. Then for every $u \in C_0^\infty(K)$ we have*

$$(1.4) \quad \frac{\mathcal{C} s^p}{1-s} \int_K \frac{|u|^p}{d_K^{s p}} dx \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+s p}} dx dy,$$

for an explicit constant $\mathcal{C} = \mathcal{C}(N, p) > 0$ (see Remark 4.1 below).

The proof of Theorem 1.1 is based on the fact that for every $0 < s < 1$ and $1 < p < \infty$ we have in weak sense

$$(-\Delta_p)^s d_K^s \geq 0 \quad \text{in } K,$$

where $(-\Delta_p)^s$ is the *fractional p -Laplacian of order s* . In other words, the function d_K^s is (s, p) -superharmonic in the following sense (see Proposition 3.2 below)

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|d_K(x)^s - d_K(y)^s|^{p-2} (d_K(x)^s - d_K(y)^s) (\varphi(x) - \varphi(y))}{|x - y|^{N+s p}} dx dy \geq 0,$$

for every nonnegative and smooth function φ , with compact support in K . Then we will test this inequality with $\varphi = |u|^p/d_K^{s(p-1)}$.

As in the local case, this trick is an essential feature in order to prove BMO regularity of the logarithm of positive supersolutions to the fractional p -Laplacian. This in turn is a crucial step in the proof of Hölder continuity of solutions to equations involving $(-\Delta_p)^s$. In this respect, this idea has already been exploited by Di Castro, Kuusi and Palatucci in [7, Lemma 1.3] (see also [12, Lemma 3.4] for the case $p = 2$). However, we observe that the computations in [7, Lemma 1.3] do not lead to the desired Hardy inequality, due to a lack of symmetry in x and y . For this, we need finer algebraic manipulations and a subtler pointwise inequality: these are contained in Lemma A.5, which is one of the main

ingredient of the proof of Theorem 1.1. We refer to Remark A.6 below for a more detailed discussion on this point.

Remark 1.2 (Comparison with known results). The quest for fractional Hardy inequalities is certainly not new. In this respect, we would like to mention that in [8, Theorem 1.1] it is proved the following version

$$(1.5) \quad C_\Omega \int_\Omega \frac{|u|^p}{d_\Omega^{sp}} dx \leq \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy, \quad \text{for every } u \in C_0^\infty(\Omega),$$

under suitable assumptions on the open Lipschitz set $\Omega \subset \mathbb{R}^N$ and some restrictions on the product sp , see also [9, Corollary 3].

Observe that in the right-hand side of (1.5), the fractional Sobolev seminorm is now computed on $\Omega \times \Omega$, rather than on the whole $\mathbb{R}^N \times \mathbb{R}^N$. However, as pointed out in [8], such a stronger inequality *fails to hold* for $sp \leq 1$, whenever Ω is bounded.

On the other hand, when Ω is a half-space, inequality (1.5) holds for $sp \neq 1$. In this case, the sharp constant has been computed by Bogdan and Dyda in [1, Theorem 1] for $p = 2$ and by Frank and Seiringer in [10, Theorem 1.1] for a general $1 < p < \infty$. We also mention that when $sp > 1$ and $\Omega \subset \mathbb{R}^N$ is an open convex set, inequality (1.5) with sharp constant (which is the same as in the half-space) has been proved by Loss and Sloane in [13, Theorem 1.2].

We point out that our proof is different from that of the aforementioned results and our Hardy inequality (4.6) holds without any restriction on the product sp .

The constant obtained in Theorem 1.1 is very likely not sharp. However, a couple of comments are in order on this point.

Remark 1.3 (Asymptotic behaviour in s of the constant). We recall that if $u \in C_0^\infty(K)$, then we have (see [17, Corollary 1.3] and [4, Proposition 2.8])

$$\lim_{s \nearrow 1} (1 - s) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy = \alpha_{N,p} \int_K |\nabla u|^p dx,$$

with

$$\alpha_{N,p} = \frac{1}{p} \int_{\mathbb{S}^{N-1}} |\langle \omega, \mathbf{e}_1 \rangle|^p d\mathcal{H}^{N-1}(\omega), \quad \mathbf{e}_1 = (1, 0, \dots, 0).$$

In this respect, we observe that the constant appearing in Theorem 1.1 has the correct asymptotic behaviour as s converges to 1: by passing to the limit in (4.6) as s goes to 1, we obtain the usual local Hardy inequality

$$(1.6) \quad C \int_K \frac{|u|^p}{d_K^p} dx \leq \alpha_{N,p} \int_K |\nabla u|^p dx.$$

As for the limit $s \searrow 0$, we recall that (see [14, Theorem 3])

$$\lim_{s \searrow 0} s \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy = \beta_{N,p} \int_K |u|^p dx,$$

with

$$\beta_{N,p} = \frac{2N\omega_N}{p},$$

and ω_N is the volume of the N -dimensional unit ball. Thus, in this case, our constant in (4.6) does not exhibit the correct asymptotic behaviour as s converges to 0. However, in this case there is an easy proof of the Hardy inequality based on geometric arguments, which produces the correct behaviour of the constant as s goes to 0 (but *not* as s goes to 1, of course), see Proposition 4.2 below.

Remark 1.4 (Dependence on N of \mathcal{C}). At a first glance, it may look strange that the constant \mathcal{C} in Theorem 1.1 depends on N . Indeed, we have seen in (1.3) that in the local case such an inequality holds with the universal constant

$$\left(\frac{p-1}{p}\right)^p.$$

However, it is easily seen that \mathcal{C} *must* depend on N in the fractional case. Indeed, we have already seen that passing to the limit in (4.6) as s goes to 1, we obtain the local Hardy inequality (1.6). As we already said, the sharp constant in the previous inequality is $(p-1/p)^p$, which means that we must have

$$\mathcal{C} \leq \left(\frac{p-1}{p}\right)^p \alpha_{N,p}, \quad \text{for every } N \geq 1.$$

On the other hand, it is easily seen that $\alpha_{N,p}$ converges to 0 as N goes ∞ . This shows that \mathcal{C} in (4.6) must depend on N .

We conclude this introduction with some consequences of Theorem 1.1. For an open set $\Omega \subset \mathbb{R}^N$ we define the *homogeneous Sobolev-Slobodeckii space* $\mathcal{D}_0^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$u \mapsto \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Then Hardy inequality (4.6) implies that when $K \neq \mathbb{R}^N$ is an open convex set, the space $\mathcal{D}_0^{s,p}(K)$ is a functional one. In this case, inequality (4.6) automatically extends to functions in $\mathcal{D}_0^{s,p}(K)$.

Moreover, as a straightforward consequence of Theorem 1.1, we have the following

Corollary 1.5. *Let $1 < p < \infty$ and $0 < s < 1$. Let $K \subset \mathbb{R}^N$ be an open convex set such that*

$$R_K = \sup_{x \in K} d_K(x) < +\infty.$$

Then we have the continuous embedding $\mathcal{D}_0^{s,p}(K) \hookrightarrow L^p(K)$. Moreover, if we set

$$\lambda_{1,p}^s(K) := \inf_{u \in C_0^\infty(K)} \left\{ [u]_{W^{s,p}(\mathbb{R}^N)}^p : \int_K |u|^p dx = 1 \right\},$$

it holds

$$\frac{\mathcal{C} s^p}{1-s} R_K^{-sp} \leq \lambda_{1,p}^s(K),$$

where \mathcal{C} is the same constant as in Theorem 1.1.

The quantity R_K above is called *inradius* of K . Observe that this is the radius of the largest ball contained in K .

Remark 1.6 (Poincaré inequality for sets bounded in one direction). Let $\omega_0 \in \mathbb{S}^{N-1}$ and $\ell_1, \ell_2 \in \mathbb{R}$ with $\ell_1 < \ell_2$. For every open set $\Omega \subset \mathbb{R}^N$ (not necessarily bounded) contained in the slab $\mathcal{S} = \{x \in \mathbb{R}^N : \ell_1 < \langle x, \omega_0 \rangle < \ell_2\}$, as a consequence of Corollary 1.5, we also get

$$\frac{\mathcal{C} s^p}{1-s} \left(\frac{\ell_2 - \ell_1}{2} \right)^{-sp} \leq \lambda_{1,p}^s(\Omega).$$

Indeed, it is sufficient to observe that $\lambda_{1,p}^s(\Omega) \geq \lambda_{1,p}^s(\mathcal{S})$ and then use Corollary 1.5 for the convex set \mathcal{S} , for which $R_{\mathcal{S}} = (\ell_2 - \ell_1)/2$.

1.3. Plan of the paper. We start with Section 2, containing the main notations, definitions and some technical results. In this part, the main point is Proposition 2.5. In Section 3 we show that, in a convex set K , the distance function d_K raised to the power s is (s, p) -superharmonic, see Proposition 3.2. Finally, the proof of Theorem 1.1 is contained in Section 4. The paper is complemented with an Appendix, containing some pointwise inequalities which are crucially exploited in the proof of our main result.

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2. PRELIMINARIES

2.1. Notations. For $x_0 \in \mathbb{R}^N$ and $R > 0$, we use the standard notation

$$B_R(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < R\}.$$

For notational simplicity, for every $1 < p < \infty$ we introduce the function $J_p : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$J_p(t) = |t|^{p-2} t, \quad \text{for } t \in \mathbb{R}.$$

We also set

$$L_{sp}^{p-1}(\mathbb{R}^N) = \left\{ u \in L_{\text{loc}}^{p-1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} dx < +\infty \right\}.$$

If $\Omega \subset \mathbb{R}^N$ is an open set, for every $1 < p < \infty$ and $0 < s < 1$, we define

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < +\infty\},$$

where

$$[u]_{W^{s,p}(\Omega)} = \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

The local version $W_{\text{loc}}^{s,p}(\Omega)$ is defined in the usual way.

2.2. Functional analytic facts. We start with the following

Definition 2.1. Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that $u \in W_{\text{loc}}^{s,p}(\Omega) \cap L_{sp}^{p-1}(\mathbb{R}^N)$ is:

- *locally weakly (s, p) -superharmonic* in Ω if

$$(2.1) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \geq 0,$$

for every nonnegative $\varphi \in W^{s,p}(\Omega)$ with compact support in Ω ;

- *locally weakly (s, p) -subharmonic* in Ω if $-u$ is (s, p) -superharmonic in Ω ;
- *locally weakly (s, p) -harmonic* in Ω if it is both (s, p) -superharmonic and (s, p) -subharmonic.

We observe that thanks to the assumptions on u , the double integral in (2.1) is finite for every admissible test function.

The following simple result is quite standard, we omit the proof.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded measurable set. Then for every $u \in L_{sp}^{p-1}(\mathbb{R}^N)$ and $r > 0$ we have*

$$\sup_{x \in \Omega} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(y)|^{p-1}}{|x - y|^{N+sp}} dy < +\infty.$$

The following technical result will be used in the next section.

Lemma 2.3. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Given $u \in W_{\text{loc}}^{s,p}(\Omega) \cap L_{sp}^{p-1}(\mathbb{R}^N)$, $\varphi \in L^p$ with compact support in Ω and $\varepsilon > 0$, the function*

$$(x, y) \mapsto \frac{J_p(u(x) - u(y))}{|x - y|^{N+sp}} \varphi(x),$$

is summable on $\mathcal{T}_\varepsilon := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \geq \varepsilon\}$.

Proof. Let us call \mathcal{O} the support of φ . We have

$$\begin{aligned}
& \iint_{\mathcal{T}_\varepsilon} \left| \frac{J_p(u(x) - u(y))}{|x - y|^{N+sp}} \varphi(x) \right| dx dy \\
&= \iint_{\{(x,y) \in \mathcal{O} \times \mathbb{R}^N : |x-y| \geq \varepsilon\}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+sp}} |\varphi(x)| dx dy \\
&\leq \varepsilon^{-\frac{N}{p}-s} \iint_{\{(x,y) \in \mathcal{O} \times \mathcal{O} : |x-y| \geq \varepsilon\}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{\frac{N+sp}{p'}}} |\varphi(x)| dx dy \\
&+ C \iint_{\{(x,y) \in \mathcal{O} \times (\mathbb{R}^N \setminus \mathcal{O}) : |x-y| \geq \varepsilon\}} \frac{|u(x)|^{p-1} + |u(y)|^{p-1}}{|x - y|^{N+sp}} |\varphi(x)| dx dy \\
&\leq \varepsilon^{-\frac{N}{p}-s} |\mathcal{O}|^{\frac{1}{p}} \left(\iint_{\mathcal{O} \times \mathcal{O}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p'}} \left(\int_{\mathcal{O}} |\varphi|^p dx \right)^{\frac{1}{p}} \\
&+ C \iint_{\{(x,y) \in \mathcal{O} \times (\mathbb{R}^N \setminus \mathcal{O}) : |x-y| \geq \varepsilon\}} \frac{|u(x)|^{p-1} + |u(y)|^{p-1}}{|x - y|^{N+sp}} |\varphi(x)| dx dy.
\end{aligned}$$

In order to treat the last integral, we observe that

$$\{(x, y) \in \mathcal{O} \times (\mathbb{R}^N \setminus \mathcal{O}) : |x - y| \geq \varepsilon\} \subset \{(x, y) \in \mathcal{O} \times \mathbb{R}^N : |x - y| \geq \varepsilon\}.$$

Thus we obtain

$$\begin{aligned}
& \iint_{\{(x,y) \in \mathcal{O} \times (\mathbb{R}^N \setminus \mathcal{O}) : |x-y| \geq \varepsilon\}} \frac{|u(x)|^{p-1} + |u(y)|^{p-1}}{|x - y|^{N+sp}} |\varphi(x)| dx dy \\
&\leq \int_{\mathcal{O}} \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)|^{p-1} |\varphi(x)|}{|x - y|^{N+sp}} dy \right) dx \\
&+ \int_{\mathcal{O}} \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y)|^{p-1} |\varphi(x)|}{|x - y|^{N+sp}} dy \right) dx \\
&\leq \frac{N \omega_N}{sp} \varepsilon^{-sp} \left(\int_{\mathcal{O}} |u|^p dx \right)^{\frac{1}{p'}} \left(\int_{\mathcal{O}} |\varphi|^p dx \right)^{\frac{1}{p}} \\
&+ \int_{\mathcal{O}} \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y)|^{p-1}}{|x - y|^{N+sp}} dy \right) |\varphi|^p dx.
\end{aligned}$$

We conclude by observing that

$$\sup_{x \in \mathcal{O}} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y)|^{p-1}}{|x - y|^{N+sp}} dy < +\infty,$$

thanks to the fact that $u \in L_{sp}^{p-1}(\mathbb{R}^N)$, see Lemma 2.2. \square

In order to use a Moser–type argument for the proof of Theorem 1.1, we will need the following result to guarantee that a certain test function is admissible.

Lemma 2.4. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For every $u \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω and $v \in W_{\text{loc}}^{s,p}(\Omega) \cap L^\infty(\Omega)$, we have*

$$uv \in W^{s,p}(\Omega).$$

Proof. We start by observing that with simple manipulations we have

$$\begin{aligned} [uv]_{W^{s,p}(\Omega)}^p &\leq 2^{p-1} \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p |v(x)|^p}{|x - y|^{N+sp}} dx dy \\ &\quad + 2^{p-1} \iint_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p |u(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\leq 2^{p-1} \|v\|_{L^\infty(\Omega)}^p [u]_{W^{s,p}(\Omega)}^p + 2^{p-1} \iint_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p |u(y)|^p}{|x - y|^{N+sp}} dx dy. \end{aligned}$$

In order to estimate the last integral, we set $\mathcal{O} = \text{supp}(u)$ and then take \mathcal{O}' such that $\mathcal{O} \Subset \mathcal{O}' \Subset \Omega$. We then obtain

$$\begin{aligned} \iint_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p |u(y)|^p}{|x - y|^{N+sp}} dx dy &= \iint_{\mathcal{O}' \times \mathcal{O}'} \frac{|v(x) - v(y)|^p |u(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\quad + \iint_{(\Omega \setminus \mathcal{O}') \times \mathcal{O}} \frac{|v(x) - v(y)|^p |u(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\leq \|u\|_{L^\infty(\Omega)}^p [v]_{W^{s,p}(\mathcal{O}')}^p \\ &\quad + \frac{2^p |\Omega| |\mathcal{O}|}{\text{dist}(\mathcal{O}, \Omega \setminus \mathcal{O}')^{N+sp}} \|u\|_{L^\infty(\Omega)}^p \|v\|_{L^\infty(\Omega)}^p. \end{aligned}$$

This gives the desired conclusion. \square

2.3. An expedient estimate for convex sets. The following expedient result is a sort of fractional counterpart of the identity

$$|\nabla d_K| = 1 \quad \text{almost everywhere in } K.$$

As explained in the Introduction, in the local case this is an essential ingredient in the proof of the Hardy inequality for convex sets. This will play an important role in our case as well.

Proposition 2.5. *Let $K \subset \mathbb{R}^N$ be an open bounded convex set. Then we have*

$$\int_{\{y \in K : d_K(y) \leq d_K(x)\}} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy \geq \frac{C_1}{1-s} d_K(x)^{p(1-s)}, \quad \text{for a. e. } x \in K,$$

where $C_1 = C_1(N, p) > 0$ is the constant

$$C_1 = \frac{1}{p} \sup_{0 < \sigma < 1} \left[\sigma^p \mathcal{H}^{N-1} \left(\{ \omega \in \mathbb{S}^{N-1} : \langle \omega, \mathbf{e}_1 \rangle > \sigma \} \right) \right].$$

Proof. We set for simplicity $\delta = d_K(x)$, thus $B_\delta(x) \subset K$ and we have

$$\int_{\{y \in K : d_K(y) \leq d_K(x)\}} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy \geq \int_{\{y \in B_\delta(x) : d_K(y) \leq d_K(x)\}} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy.$$

We now take $x' \in \partial K$ such that $|x - x'| = \delta$. For a given $0 < \sigma < 1$, we consider the portion $\Sigma_\sigma(x)$ of $B_\delta(x)$ defined by

$$\Sigma_\sigma(x) = \left\{ y \in B_\delta(x) : \left\langle \frac{y-x}{|y-x|}, \frac{x'-x}{|x'-x|} \right\rangle > \sigma \right\},$$

see Figure 1.

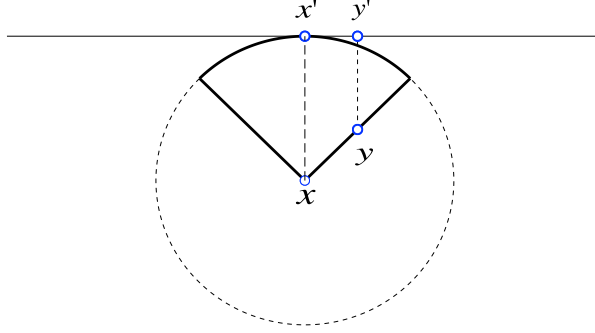


FIGURE 1. The set $\Sigma_\sigma(x)$ and the supporting hyperplane $\Pi_{x'}$.

By convexity of K , it is not difficult to see that

$$(2.2) \quad \Sigma_\sigma(x) \subset \{y \in B_\delta(x) : d_K(y) \leq d_K(x)\}, \quad \text{for every } 0 < \sigma < 1.$$

We can be more precise on this point. We denote by $\Pi_{x'}$ the supporting hyperplane of K at the point x' , orthogonal to $x' - x$. Then for every $y \in K$, we denote by y' the orthogonal projection of y on $\Pi_{x'}$. Thus by convexity we have

$$d_K(y) \leq |y - y'|, \quad \text{for every } y \in K.$$

We then observe that for every $y \in \Sigma_\sigma(x)$, it holds

$$(2.3) \quad \begin{aligned} d_K(x) = |x - x'| &= |y - y'| + \left\langle \frac{y-x}{|y-x|}, \frac{x'-x}{|x'-x|} \right\rangle |y-x| \\ &\geq d_K(y) + \sigma |y-x|. \end{aligned}$$

By using (2.2) and (2.3), we thus obtain

$$\begin{aligned}
 & \int_{\{y \in K : d_K(y) \leq d_K(x)\}} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy \\
 & \geq \int_{\Sigma_\sigma(x)} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy \\
 & \geq \sigma^p \int_{\Sigma_\sigma(x)} |x - y|^{p(1-s)-N} dy \\
 & = \sigma^p \left(\int_{\{\omega \in \mathbb{S}^{N-1} : \langle \omega, \frac{x' - x}{|x' - x|} \rangle > \sigma\}} d\mathcal{H}^{N-1}(\omega) \right) \int_0^\delta \varrho^{-1+p(1-s)} d\varrho \\
 & = \frac{f(\sigma) \sigma^p}{(1-s)p} \delta^{p(1-s)},
 \end{aligned}$$

where $f(\sigma) > 0$ is the quantity

$$f(\sigma) = \int_{\{\omega \in \mathbb{S}^{N-1} : \langle \omega, \mathbf{e}_1 \rangle > \sigma\}} d\mathcal{H}^{N-1}(\omega).$$

By arbitrariness of $0 < \sigma < 1$, we can take the supremum and get the conclusion. \square

3. SUPERHARMONICITY OF THE DISTANCE FUNCTION

In this section we will prove that d_K^s in a convex set K is weakly (s, p) -superharmonic, see Definition 2.1. We start with the case of the half-space. The proof of the following fact can be found in [11, Lemma 3.2].

Lemma 3.1. *We set $\mathbb{H}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$. Let $1 < p < \infty$ and $0 < s < 1$, then $d_{\mathbb{H}_+^N}^s$ is locally weakly (s, p) -harmonic in \mathbb{H}_+^N . Moreover, there holds*

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_{\mathbb{H}_+^N}(x)^s - d_{\mathbb{H}_+^N}(y)^s)}{|x - y|^{N+sp}} dy = 0,$$

strongly in $L_{\text{loc}}^1(\mathbb{H}_+^N)$.

By appealing to the previous result and using the geometric properties of convex sets, we can prove the following

Proposition 3.2. *Let $K \subset \mathbb{R}^N$ be an open bounded convex set. For $1 < p < \infty$ and $0 < s < 1$, we have that d_K^s is locally weakly (s, p) -superharmonic.*

Proof. We first observe that $d_K^s \in W_{\text{loc}}^{s,p}(K) \cap L_{sp}^{p-1}(\mathbb{R}^N)$. Let $\varphi \in W^{s,p}(K)$ be a nonnegative function with compact support in K , we observe that the function

$$(x, y) \mapsto \frac{J_p(d_K(x)^s - d_K(y)^s) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}},$$

is summable. Then by the Dominated Convergence Theorem, we have

$$\begin{aligned} & \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(d_K(x)^s - d_K(y)^s) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &= \lim_{\varepsilon \rightarrow 0} \iint_{\mathcal{T}_\varepsilon} \frac{J_p(d_K(x)^s - d_K(y)^s) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy, \end{aligned}$$

where we set $\mathcal{T}_\varepsilon = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \geq \varepsilon\}$. By Lemma 2.3 we have that for every fixed ε , the function

$$(x, y) \mapsto \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} \varphi(x),$$

is summable on \mathcal{T}_ε . Thus we get

$$\begin{aligned} & \iint_{\mathcal{T}_\varepsilon} \frac{J_p(d_K(x)^s - d_K(y)^s) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &= \iint_{\mathcal{T}_\varepsilon} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} \varphi(x) dx dy - \iint_{\mathcal{T}_\varepsilon} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} \varphi(y) dx dy \\ &= 2 \int_K \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} dy \right) \varphi(x) dx. \end{aligned}$$

In the second equality, we used Fubini's Theorem and the fact that φ has compact support contained in K . In order to conclude, we need to show that

$$(3.1) \quad \lim_{\varepsilon \rightarrow 0} \int_K \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} dy \right) \varphi(x) dx \geq 0.$$

We now take $x \in K$ and $0 < \varepsilon < d_K(x)$, then we consider a point $x' \in \partial K$ such that $d_K(x) = |x - x'|$. We take a supporting hyperplane to K at x' , up to a rigid motion we can suppose that this is given by $\{x \in \mathbb{R}^N : x_N = 0\}$ and that $K \subset \mathbb{H}_+^N$. We observe that by convexity of K

$$d_K(y) \leq d_{\mathbb{H}_+^N}(y), \quad \text{for } y \in \mathbb{R}^N,$$

and

$$d_K(x) = |x - x'| = d_{\mathbb{H}_+^N}(x),$$

see Figure 2.

By exploiting these facts and the monotonicity of J_p , we obtain for almost every $x \in K$

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} dy \geq \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_{\mathbb{H}_+^N}(x)^s - d_{\mathbb{H}_+^N}(y)^s)}{|x - y|^{N+sp}} dy = 0,$$

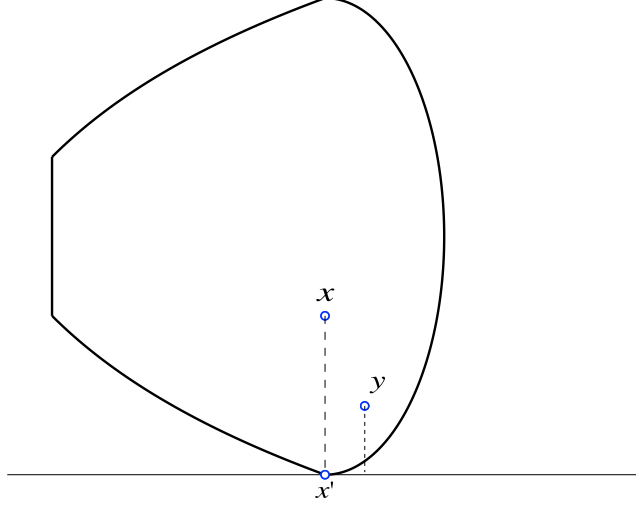


FIGURE 2. The distance of y from ∂K is smaller than its distance from the hyperplane.

where the last equality follows from Lemma 3.1. By multiplying the previous inequality by φ nonnegative, integrating over K and using Fatou's Lemma, we thus obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_K \left(\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} dy \right) \varphi(x) dx \\ & \geq \int_K \left(\liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} dy \right) \varphi(x) dx \geq 0. \end{aligned}$$

This proves (3.1) and thus we get the desired conclusion. \square

4. PROOF OF HARDY INEQUALITY

4.1. Proof of Theorem 1.1. We divide the proof in two cases: first we prove the result under the additional assumptions that K is bounded, then we extend it to general convex sets not coinciding with the whole space.

Case 1: bounded convex sets. By Proposition 3.2 we know that

$$(4.1) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(d_K(x)^s - d_K(y)^s)(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \geq 0,$$

for every nonnegative $\varphi \in W^{s,p}(K)$ with compact support in K . Then we test with

$$\varphi = \frac{|u|^p}{(d_K^s + \varepsilon)^{p-1}},$$

where $u \in C_0^\infty(K)$ and $\varepsilon > 0$. By Lemma 2.4, we have that φ is admissible. Indeed, we already know that $d_K^s \in W_{\text{loc}}^{s,p}(K) \cap L^\infty(K)$. Moreover, for every $\varepsilon > 0$ the function

$f(t) = (t + \varepsilon)^{1-p}$ is Lipschitz for $t > 0$, thus $(d_K^s + \varepsilon)^{1-p} = f \circ d_K^s \in W_{\text{loc}}^{s,p}(K) \cap L^\infty(K)$ as well.

Let us call \mathcal{O} the support of u , then from (4.1) we have

$$(4.2) \quad \begin{aligned} 0 \leq & \iint_{K \times K} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} \left(\frac{|u(x)|^p}{(d_K(x)^s + \varepsilon)^{p-1}} - \frac{|u(y)|^p}{(d_K(y)^s + \varepsilon)^{p-1}} \right) dx dy \\ & + 2 \iint_{\mathcal{O} \times (\mathbb{R}^N \setminus K)} \frac{J_p(d_K(x)^s)}{|x - y|^{N+sp}} \frac{|u(x)|^p}{(d_K(x)^s + \varepsilon)^{p-1}} dx dy. \end{aligned}$$

We first observe that

$$(4.3) \quad \iint_{\mathcal{O} \times (\mathbb{R}^N \setminus K)} \frac{J_p(d_K(x)^s)}{|x - y|^{N+sp}} \frac{|u(x)|^p}{(d_K(x)^s + \varepsilon)^{p-1}} dx dy \leq \iint_{\mathcal{O} \times (\mathbb{R}^N \setminus K)} \frac{|u(x)|^p}{|x - y|^{N+sp}} dx dy.$$

We now need to estimate the double integral

$$\mathcal{I} := \iint_{K \times K} \frac{J_p(d_K(x)^s - d_K(y)^s)}{|x - y|^{N+sp}} \left(\frac{|u(x)|^p}{(d_K(x)^s + \varepsilon)^{p-1}} - \frac{|u(y)|^p}{(d_K(y)^s + \varepsilon)^{p-1}} \right) dx dy.$$

For this, we crucially exploit the fundamental inequality of Lemma A.5, with the choices

$$a = d_K(x)^s + \varepsilon, \quad b = d_K(y)^s + \varepsilon, \quad c = |u(x)|, \quad d = |u(y)|.$$

This entails

$$(4.4) \quad \begin{aligned} \mathcal{I} \leq & -C_2 \iint_{K \times K} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s + 2\varepsilon} \right|^p (|u(x)|^p + |u(y)|^p) \frac{dx dy}{|x - y|^{N+sp}} \\ & + C_3 \iint_{K \times K} \frac{||u(x)| - |u(y)||^p}{|x - y|^{N+sp}} dx dy, \end{aligned}$$

where C_2 and C_3 are as in Lemma A.5. By using (4.4) in (4.2), together with (4.3), we obtain

$$(4.5) \quad C_2 \iint_{K \times K} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s} \right|^p (|u(x)|^p + |u(y)|^p) \frac{dx dy}{|x - y|^{N+sp}} \leq C_3 [\|u\|_{W^{s,p}(\mathbb{R}^N)}^p].$$

To obtain (4.5), we also took the limit as ε goes to 0 and used Fatou's Lemma. We observe that by symmetry, we have

$$\begin{aligned} & \iint_{K \times K} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s} \right|^p (|u(x)|^p + |u(y)|^p) \frac{dx dy}{|x - y|^{N+sp}} \\ & = 2 \iint_{K \times K} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s} \right|^p |u(x)|^p \frac{dx dy}{|x - y|^{N+sp}} \\ & \geq 2 \int_K \left(\int_{\{y \in K : d_K(y) \leq d_K(x)\}} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s} \right|^p \frac{dy}{|x - y|^{N+sp}} \right) |u(x)|^p dx. \end{aligned}$$

We now use the pointwise inequality (A.2), so to obtain

$$\begin{aligned} & \iint_{K \times K} \left| \frac{d_K(x)^s - d_K(y)^s}{d_K(x)^s + d_K(y)^s} \right|^p (|u(x)|^p + |u(y)|^p) \frac{dx dy}{|x - y|^{N+sp}} \\ & \geq \frac{s^p}{2^{p-1}} \int_K \left(\int_{\{y \in K : d_K(y) \leq d_K(x)\}} \frac{|d_K(x) - d_K(y)|^p}{|x - y|^{N+sp}} dy \right) \frac{|u(x)|^p}{d_K(x)^p} dx. \end{aligned}$$

By using this in (4.5) and then applying the expedient estimate of Proposition 2.5, we end up with

$$\frac{s^p \mathcal{C}}{1-s} \int_K \frac{|u|^p}{d_K^{sp}} dx \leq [u]_{W^{s,p}(\mathbb{R}^N)}^p,$$

where we have used the triangle inequality to replace the seminorm of $|u|$ with that of u . This concludes the proof.

Case 2: general convex sets. We now take $K \neq \mathbb{R}^N$ an open unbounded convex sets. For every $R > 0$ we set $K_R = K \cap B_R(0)$. Let us take $u \in C_0^\infty(K)$, then for every R large enough, we have $u \in C_0^\infty(K_R)$ as well. By using the previous case, we then get

$$\frac{s^p \mathcal{C}}{1-s} \int_K \frac{|u|^p}{d_{K_R}^{sp}} dx = \frac{s^p \mathcal{C}}{1-s} \int_{K_R} \frac{|u|^p}{d_{K_R}^{sp}} dx \leq [u]_{W^{s,p}(\mathbb{R}^N)}^p.$$

By observing that $d_{K_R} \leq d_K$, we then get the desired conclusion.

Remark 4.1. A closer inspection of the proof of Theorem 1.1 reveals that the constant \mathcal{C} appearing in (4.6) is given by

$$\mathcal{C} = \frac{C_1 C_2}{2^{p-1} C_3},$$

where $C_1 = C_1(N, p)$ is the constant of Proposition 2.5, and C_2, C_3 (which depend only on p) come from Lemma A.5.

4.2. Improved constant for s close to 0. By means of elementary geometric considerations, we can prove a Hardy inequality with a constant having the correct asymptotic behaviour as s goes to 0. This is the content of the next result, whose proof is essentially contained in [5, pages 440–441], as pointed out to us by Bartłomiej Dyda.

Proposition 4.2. *Let $1 < p < \infty$ and $0 < s < 1$. Let $K \subset \mathbb{R}^N$ be an open convex set such that $K \neq \mathbb{R}^N$. Then for every $u \in C_0^\infty(K)$ we have*

$$(4.6) \quad \frac{C}{s} \int_K \frac{|u|^p}{d_K^{sp}} dx \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy,$$

for an explicit constant $C = C(N, p) > 0$.

Proof. For every $x \in K$, we take $x_0 \in \mathbb{R}^N \setminus K$ such that $|x - x_0| = 2d_K(x)$. Then we can estimate

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \geq \int_K |u(x)|^p \left(\int_{(\mathbb{R}^N \setminus K) \setminus B_{d_K(x)}(x_0)} \frac{dy}{|x - y|^{N+sp}} \right) dx.$$

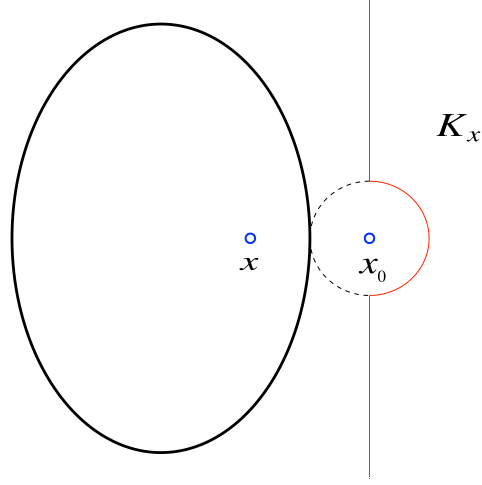


FIGURE 3. The set K_x in the proof of Proposition 4.2.

We observe that for every $y \in \mathbb{R}^N \setminus B_{d_K(x)}(x_0)$, we have

$$|x - y| \leq |x - x_0| + |x_0 - y| = 2d_K(x) + |x_0 - y| \leq 3|x_0 - y|.$$

By convexity, we have that (see Figure 3)

$$K_x := \{y \in \mathbb{R}^N \setminus B_{d_K(x)}(x_0) : \langle y - x_0, x - x_0 \rangle < 0\} \subset (\mathbb{R}^N \setminus K) \setminus B_{d_K(x)}(x_0).$$

By joining the last two informations, we get

$$\begin{aligned} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy &\geq \frac{1}{3^{N+sp}} \int_K |u(x)|^p \left(\int_{K_x} \frac{dy}{|x_0 - y|^{N+sp}} \right) dx \\ &= \frac{1}{3^{N+sp}} \left(\int_{\{\omega \in \mathbb{S}^{N-1} : \langle \omega, \mathbf{e}_1 \rangle < 0\}} d\mathcal{H}^{N-1} \right) \int_{d_K(x)}^{+\infty} \varrho^{-1-sp} d\varrho \\ &\quad \times \int_K |u(x)|^p dx \\ &= \frac{N \omega_N}{2 \cdot 3^{N+sp} \cdot p} \frac{1}{s} \int_K |u|^p dx. \end{aligned}$$

This concludes the proof. \square

APPENDIX A. SOME POINTWISE INEQUALITIES

We collect here some pointwise inequalities needed throughout the whole paper. The most important one is Lemma A.5. We recall the notation

$$J_p(t) = |t|^{p-2} t, \quad \text{for } t \in \mathbb{R}.$$

Lemma A.1. *Let $1 < p < \infty$, for every $a, b > 0$ we have*

$$J_p(a-b) \left(\frac{1}{b^{p-1}} - \frac{1}{a^{p-1}} \right) \geq (p-1) |\log b - \log a|^p.$$

Equality holds if and only if $a = b$.

Proof. This is proved in [3, Lemma A.2 & Remark A.3]. □

Lemma A.2. *For every $a, b > 0$ we have*

$$\frac{|a-b|}{a+b} \leq |\log a - \log b|.$$

Equality holds if and only if $a = b$.

Proof. We observe that if $a = b$ there is nothing to prove. We then take $a \neq b$ and without loss of generality we can suppose $a > b$. The seeked inequality is then equivalent to

$$\frac{a-b}{a+b} \leq \log \frac{a}{b}, \quad \text{for } 0 < b < a.$$

By setting $t = b/a$, this in turn is equivalent to prove that

$$\frac{1-t}{1+t} \leq -\log t, \quad \text{for } 0 < t < 1.$$

By basic Calculus, it is easily seen that the function

$$\varphi(t) = \log t + \frac{1-t}{1+t},$$

is strictly increasing for $t \in (0, 1)$ and $\varphi(1) = 0$. This gives the desired conclusion. □

Remark A.3. By combining Lemma A.1 and A.2, we also obtain

$$(A.1) \quad J_p(a-b) \left(\frac{1}{b^{p-1}} - \frac{1}{a^{p-1}} \right) \geq (p-1) \left| \frac{a-b}{a+b} \right|^p,$$

for every $a, b > 0$ and $1 < p < \infty$.

Lemma A.4. *Let $0 < s < 1$, then for every $a, b > 0$ we have*

$$(A.2) \quad \frac{|a^s - b^s|}{a^s + b^s} \geq \frac{s}{2} \frac{|a-b|}{\max\{a, b\}}.$$

Proof. For $a = b$ there is nothing to prove. Without loss of generality, we can assume $a > b$. By defining $t = b/a \in (0, 1)$, inequality (A.2) is equivalent to prove

$$\frac{1-t^s}{1+t^s} \geq \frac{s}{2} (1-t).$$

We observe that by the “below tangent” property of concave functions, we have

$$t^s \leq 1 + s(t-1) \quad \text{i.e.} \quad 1 - t^s \geq s(1-t).$$

By combining this with the trivial estimate $1 + t^s \leq 2$, we get the conclusion. □

An essential ingredient in the proof of our main result has been the following pointwise inequality.

Lemma A.5 (Fundamental inequality). *Let $1 < p < \infty$ and let $a, b, c, d \in \mathbb{R}$, with $a, b > 0$ and $c, d \geq 0$. Then there exist two constants $C_2 = C_2(p) > 0$ and $C_3 = C_3(p) > 1$, such that*

$$(A.3) \quad J_p(a-b) \left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right) + C_2 \left| \frac{a-b}{a+b} \right|^p (c^p + d^p) \leq C_3 |c-d|^p.$$

Proof. We observe that for $a = b$ there is nothing to prove, since the left-hand side vanishes. Without loss of generality, we can assume $a > b$. Also notice that if $c \leq d$, then

$$\begin{aligned} (a-b)^{p-1} \left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right) &\leq (a-b)^{p-1} \left(\frac{d^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right) \\ &= -d^p (a-b)^{p-1} \left(\frac{1}{b^{p-1}} - \frac{1}{a^{p-1}} \right) \\ &\leq -(p-1) d^p \left| \frac{a-b}{a+b} \right|^p \leq -(p-1) \frac{c^p + d^p}{2} \left| \frac{a-b}{a+b} \right|^p, \end{aligned}$$

where in the second inequality we used (A.1). Thus inequality (A.3) holds with $C_2 = (p-1)/2$ and $C_3 > 0$ arbitrary.

We assume now that $a > b$ and $c > d$, then by setting

$$t = b/a \in (0, 1) \quad \text{and} \quad A = d/c \in [0, 1),$$

inequality (A.3) is equivalent to

$$(A.4) \quad (1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right) + C_2 \left(\frac{1-t}{1+t} \right)^p (1+A^p) \leq C_3 (1-A)^p,$$

with $t \in (0, 1)$ and $A \in [0, 1)$. We study the function

$$(A.5) \quad \Phi(t) = (1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right), \quad t \in (0, 1),$$

which is maximal for $t = A$. This in particular implies¹

$$(A.6) \quad (1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right) \leq (1-A)^p.$$

We now distinguish two cases:

$$\text{either} \quad 0 \leq A \leq \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < A < 1.$$

¹We observe that this is equivalent to

$$J_p(a-b) \left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right) \leq |c-d|^p,$$

which is a discrete version of *Picone's inequality*, see [2, Proposition 4.2].

A. Case $0 \leq A \leq 1/2$. This is the simplest case. Indeed, we have

$$\frac{1-t}{1+t} \leq 1 \quad \text{and} \quad (1-A)^p \geq \frac{1}{2^p}.$$

Thus by using this and (A.6), we get

$$(1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}}\right) + \frac{C_3 - 1}{2^{p+1}} \left(\frac{1-t}{1+t}\right)^p (1+A^p) \leq C_3 (1-A)^p,$$

which is (A.4) with $C_2 = (C_3 - 1)/2^{p+1}$ and $C_3 > 1$ arbitrary.

B. Case $1/2 < A < 1$. Here in turn we consider two subcases: $A \leq t$ and $0 < t < A$.

B.1. Case $1/2 < A < 1$ and $t \geq A$. This is easy, since we directly have

$$(1-t)^p \leq (1-A)^p,$$

and thus

$$\left(\frac{1-t}{1+t}\right)^p \leq (1-t)^p \leq (1-A)^p.$$

By using this and (A.6), we get

$$(1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}}\right) + \frac{C_3 - 1}{2} \left(\frac{1-t}{1+t}\right)^p (1+A^p) \leq C_3 (1-A)^p,$$

which is (A.3) with $C_2 = (C_3 - 1)/2$ and $C_3 > 1$ arbitrary.

B.1. Case $1/2 < A < 1$ and $0 < t < A$. Here we need to study in more details the function Φ defined in (A.5). We have

$$\begin{aligned} \Phi''(t) &= (p-1)(1-t)^{p-3} \left[p-2 + 2\frac{A^p}{t^p} - p\frac{A^p}{t^{p+1}} \right] \\ &= (p-1)(1-t)^{p-2} \left[\frac{p-2}{1-t} \left(1 - \frac{A^p}{t^p}\right) - p\frac{A^p}{t^{p+1}} \right], \quad t \in (0, 1). \end{aligned}$$

By an easy computation, we can see that

$$t \mapsto p-2 + 2\frac{A^p}{t^p} - p\frac{A^p}{t^{p+1}}$$

is monotone increasing, thus we get

$$\Phi''(t) \leq (p-1)(1-t)^{p-3} p \left(1 - \frac{1}{A}\right), \quad \text{for } 0 < t < A.$$

In particular, we get that Φ is concave on the interval $(0, A)$. We use a second order Taylor expansion around the maximum point $t = A$, i.e.

$$\begin{aligned}
\Phi(t) &= \Phi(A) + \int_t^A \Phi''(s) (s - t) ds \\
\text{(A.7)} \quad &= (1 - A)^p \\
&\quad + (p - 1) \int_t^A (1 - s)^{p-2} \left[\frac{p-2}{1-s} \left(1 - \frac{A^p}{s^p} \right) - p \frac{A^p}{s^{p+1}} \right] (s - t) ds,
\end{aligned}$$

where we used that $\Phi'(A) = 0$. In order to estimate the remainder term inside the integral, we distinguish once again two cases:

- if $1 < p \leq 2$, then by using Lemma A.7 below and the fact that $A \geq 1/2$, we get

$$\begin{aligned}
&\int_t^A (1 - s)^{p-2} \left[\frac{p-2}{1-s} \left(1 - \frac{A^p}{s^p} \right) - p \frac{A^p}{s^{p+1}} \right] (s - t) ds \\
&= \int_t^A \left[(p-2) \frac{s^p - A^p}{1-s} - p \frac{A^p}{s} \right] \frac{(1-s)^{p-2}}{s^p} (s - t) ds \\
&\leq -\frac{p(p-1)}{2^p} \int_t^A \frac{(1-s)^{p-2}}{s^p} (s - t) ds \\
&\leq -\frac{p(p-1)}{2^p} (1-t)^{p-2} \int_t^A (s - t) ds \\
&= -\frac{p(p-1)}{2^{p+1}} (1-t)^{p-2} (A-t)^2.
\end{aligned}$$

By using the previous estimate in (A.7), we have

$$\text{(A.8)} \quad (1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right) \leq (1-A)^p - \frac{p(p-1)^2}{2^{p+1}} (1-t)^{p-2} (A-t)^2.$$

It is now sufficient to observe that

$$\begin{aligned}
(1-t)^p &= (1-t)^{p-2} (1-t)^2 \\
&\leq (1-t)^{p-2} \left(2(A-t)^2 + 2(1-A)^2 \right) \\
&\leq 2(1-t)^{p-2} (A-t)^2 + 2(1-A)^p
\end{aligned}$$

and thus

$$\text{(A.9)} \quad C_2 \left(\frac{1-t}{1+t} \right)^p (1+A^p) \leq 4C_2(1-t)^{p-2} (A-t)^2 + 4C_2(1-A)^p.$$

If we sum up (A.8) and (A.9) and choose

$$C_2 = \frac{1}{4} \min \left\{ C_3 - 1, \frac{p(p-1)^2}{2^{p+1}} \right\},$$

we get again the desired conclusion (A.4), with $C_3 > 1$ arbitrary.

- if $p > 2$, then

$$\begin{aligned}
 & \int_t^A (1-s)^{p-2} \left[\frac{p-2}{1-s} \left(1 - \frac{A^p}{s^p} \right) - p \frac{A^p}{s^{p+1}} \right] (s-t) ds \\
 & \leq -p A^p \int_t^A \frac{(1-s)^{p-2}}{s^{p+1}} (s-t) ds \\
 & \leq -\frac{p}{2^p} \int_t^A (1-s)^{p-2} (s-t) ds.
 \end{aligned}$$

The last integral can be explicitly computed: we have

$$\begin{aligned}
 \int_t^A (1-s)^{p-2} (s-t) ds &= -\frac{1}{p-1} (1-A)^{p-1} (A-t) + \frac{1}{p(p-1)} (1-t)^p \\
 &\quad - \frac{1}{p(p-1)} (1-A)^p.
 \end{aligned}$$

We use Young's inequality to estimate the first term on the right-hand side

$$\begin{aligned}
 -\frac{1}{p-1} (1-A)^{p-1} (A-t) &\geq -\frac{p}{(p-1)^2} \varepsilon^{-\frac{1}{p-1}} (1-A)^p \\
 &\quad - \frac{\varepsilon}{p(p-1)} (A-t)^p,
 \end{aligned}$$

with $\varepsilon > 0$. We use these estimates in (A.7). This in turn gives

$$\begin{aligned}
 (1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right) &\leq (1-A)^p + \frac{p^2}{2^p(p-1)} \varepsilon^{-\frac{1}{p-1}} (1-A)^p \\
 &\quad + \frac{\varepsilon}{2^p} (A-t)^p \\
 &\quad - \frac{1}{2^p} (1-t)^p + \frac{1}{2^p} (1-A)^p.
 \end{aligned}$$

By choosing $\varepsilon = 1/2$ and using that $A-t \leq 1-t$, we then obtain

$$(1-t)^{p-1} \left(1 - \frac{A^p}{t^{p-1}} \right) \leq C_3 (1-A)^p - \frac{1}{2^{p+1}} (1-t)^p,$$

with

$$C_3 = 1 + \frac{p^2}{2^p(p-1)} 2^{\frac{1}{p-1}} + \frac{1}{2^p}.$$

Once again, this is enough to get the desired conclusion, since

$$C_2 \left(\frac{1-t}{1+t} \right)^p (1+A^p) \leq 2C_2 (1-t)^p.$$

Then we only need to choose $C_2 = 1/2^{p+2}$ in order to get (A.4).

We thus concluded the proof. \square

Remark A.6. Inequality (A.3) looks similar to the pointwise inequality which can be found right before [7, equation (3.12), page 1289], where the term

$$\left| \frac{a-b}{a+b} \right|^p (c^p + d^p),$$

is replaced by

$$|\log a - \log b|^p \min\{c^p, d^p\}.$$

The main difference is that our inequality is symmetric in the terms c and d , which is a crucial feature in order to prove Theorem 1.1. On the other hand, the inequality in [7] *can not* have this property and thus it is not useful in order to prove Hardy inequality.

In the previous result, we needed the following inequality in order to deal with the case $1 < p \leq 2$.

Lemma A.7. *Let $1 < p \leq 2$, for every $s \in (0, 1)$ and $A \in [0, 1]$ we have*

$$(p-2) \frac{s^p - A^p}{1-s} - p \frac{A^p}{s} \leq -p(p-1) A^p.$$

Proof. We rewrite

$$(A.10) \quad (p-2) \frac{s^p - A^p}{1-s} - p \frac{A^p}{s} = (2-p) \frac{A^p - s^p}{1-s} - p \frac{A^p}{s}.$$

We then observe that

$$(A.11) \quad \frac{A^p - s^p}{1-s} \leq p A^p.$$

Indeed, the latter is equivalent to

$$A^p(1-p+sp) \leq s^p,$$

which is easily seen to be true. By using (A.11) in (A.10), we now obtain

$$(p-2) \frac{s^p - A^p}{1-s} - p \frac{A^p}{s} \leq (2-p) p A^p - p \frac{A^p}{s} \leq -p(p-1) A^p,$$

as desired. □

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