On the existence of connecting orbits for critical values of the energy

Giorgio Fusco,[*] Giovanni F. Gronchi,[†] Matteo Novaga[‡]

January 26, 2017

Abstract

We consider an open connected set Ω and a smooth potential U which is positive in Ω and vanishes on $\partial\Omega$. We study the existence of orbits of the mechanical system

$$\ddot{u} = U_x(u),$$

that connect different components of $\partial\Omega$ and lie on the zero level of the energy. We allow that $\partial\Omega$ contains a finite number of critical points of U. The case of symmetric potential is also considered.

1 Introduction

Let $U : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2. We assume that $\Omega \subset \mathbb{R}^n$ is a connected component of the set $\{x \in \mathbb{R}^n : U(x) > 0\}$ and that $\partial\Omega$ is compact and is the union of $N \geq 1$ distinct nonempty connected components $\Gamma_1, \ldots, \Gamma_N$. We consider the following situations

H $N \geq 2$ and, if Ω is unbounded, there is $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{+\infty} \sigma(r)dr = +\infty$ and

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0. \tag{1.1}$$

$\mathbf{H}_s$ Ω is bounded, the origin $0 \in \mathbb{R}^n$ belongs to Ω and U is invariant under the antipodal map

$$U(-x) = U(x), \quad x \in \Omega.$$

Condition (1.1) was first introduced in [7]. A sufficient condition for (1.1) is that $\liminf_{|x|\to\infty} U(x) > 0$.

We study non constant solutions $u : (T_-, T_+) \to \Omega$, of the equation

$$\ddot{u} = U_x(u), \quad U_x = \Big(\frac{\partial U}{\partial x}\Big)^T, \tag{1.2}$$

that satisfy

$$\lim_{t\to T_\pm} d(u(t), \partial\Omega) = 0, \tag{1.3}$$

with d the Euclidean distance, and lie on the energy surface

$$\frac{1}{2}|\dot{u}|^2 - U(u) = 0. \tag{1.4}$$

We allow that the boundary $\partial\Omega$ of Ω contains a finite set P of critical points of U and assume

[*]Dipartimento di Matematica, Università dell'Aquila; e-mail: fusco@univaq.it
[†]Dipartimento di Matematica, Università di Pisa; e-mail: giovanni.federico.gronchi@unipi.it
[‡]Dipartimento di Matematica, Università di Pisa; e-mail: matteo.novaga@unipi.it

$\mathbf{H}_1$ If $\Gamma \in \{\Gamma_1, \dots, \Gamma_N\}$ has positive diameter and $p \in P \cap \Gamma$ then p is a hyperbolic critical point of U.

If Γ has positive diameter, then hyperbolic critical points $p \in \Gamma$ correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (1.2). These points are organizing centers of complex dynamics, see [6].
Note that $\mathbf{H}_1$ does not exclude that some of the Γ_j reduce to a singleton, say $\{p\}$, for some $p \in P$. In this case nothing is required on the behavior of U in a neighborhood of p aside from being C^2.
A comment on $\mathbf{H}$ and $\mathbf{H}_s$ is in order. If P is nonempty $u \equiv p$ for $p \in P$ is a constant solution of (1.2) that satisfies (1.3) and (1.4). To avoid trivial solutions of this kind we require $N \geq 2$ in $\mathbf{H}$, and look for solutions that connect different components of $\partial\Omega$. In $\mathbf{H}_s$ we do not exclude that $\partial\Omega$ is connected ($N = 1$) and avoid trivial solutions by restricting to a symmetric context and to solutions that pass through 0.

We prove the following results.

Theorem 1.1. *Assume that $\mathbf{H}$ and $\mathbf{H}_1$ hold. Then for each $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$ there exist $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ and a map $u^* : (T_-, T_+) \to \Omega$, with $-\infty \leq T_- < T_+ \leq +\infty$, that satisfies (1.2), (1.4) and*

$$\lim_{t \to T_\pm} d(u^*(t), \Gamma_\pm) = 0. \tag{1.5}$$

Moreover, $T_- > -\infty$ (resp. $T_+ < +\infty$) if and only if Γ_- (resp. Γ_+) has positive diameter. If $T_- > -\infty$ it results

$$\begin{aligned} &\lim_{t \to T_-} u^*(t) = x_-, \\ &\lim_{t \to T_-} \dot{u}^*(t) = 0, \end{aligned} \tag{1.6}$$

for some $x_- \in \Gamma_- \setminus P$. An analogous statement holds if $T_+ < +\infty$.

Theorem 1.2. *Assume that $\mathbf{H}_s$ and $\mathbf{H}_1$ hold. Then there exist $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$ and a map $u^* : (0, T_+) \to \Omega$, with $0 < T_+ \leq +\infty$, that satisfies (1.2), (1.4) and*

$$\lim_{t \to T_+} d(u^*(t), \Gamma_+) = 0.$$

Moreover, $T_+ < +\infty$ if and only if Γ_+ has positive diameter. If $T_+ < +\infty$ it results

$$\begin{aligned} &\lim_{t \to T_+} u^*(t) = x_+, \\ &\lim_{t \to T_+} \dot{u}^*(t) = 0, \end{aligned}$$

for some $x_+ \in \Gamma_+ \setminus P$.

We list a few straightforward consequences of Theorems 1.1 and 1.2.

Corollary 1.3. *Theorem 1.1 implies that, if $\partial\Omega = P$, given $p_- \in P$ there is $p_+ \in P \setminus \{p_-\}$ and a heteroclinic connection between p_- and p_+, that is a solution $u^* : \mathbb{R} \to \mathbb{R}^n$ of (1.2) and (1.4) that satisfies*

$$\lim_{t \to \pm\infty} u^*(t) = p_\pm.$$

The problem of the existence of heteroclinic connections between two isolated zeros $p_\pm$ of a non-negative potential has been recently reconsidered by several authors. In [1] existence was established under a mild monotonicity condition on U near $p_\pm$. This condition was removed in [8], see also [2]. The most general results, equivalent to the consequence of Theorem 1.1 discussed in Section 2.1, were recently obtained in [7] and in [11], see also [3]. All these papers establish existence by a variational approach. In [1], [8] and [2] by minimizing the action functional, and in [7] and [11] by minimizing the Jacobi functional.

Corollary 1.4. *Theorem 1.1 implies that, if $\Gamma_- = \{p\}$ for some $p \in P$ and the elements of $\{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ have all positive diameter, there exists a nontrivial orbit homoclinic to p that satisfies (1.2), (1.4).*

Proof. Let $v^* : \mathbb{R} \to \Omega \cup \{x_+\}$ be the extension defined by

$$v^*(T_+ + t) = u^*(T_+ - t), \ \ t \in (0, +\infty), \ \ v^*(T_+) = x_+,$$

of the solution $u^* : (-\infty, T_+) \to \Omega$ given by Theorem 1.1. The map v^* so defined is a smooth non-constant solution of (1.2) that satisfies

$$\lim_{t \to \pm\infty} v^*(t) = p.$$

□

Corollary 1.5. *Theorem 1.1 implies that, if all the sets $\Gamma_1, \dots, \Gamma_N$ have positive diameter, given $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$, there exist $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ and a periodic solution $v^* : \mathbb{R} \to \Omega$ of (1.2) and (1.4) that oscillates between Γ_- and Γ_+. This solution has period $T = 2(T_+ - T_-)$.*

Proof. The solution v^* is the T-periodic extension of the map $w^* : [T_-, 2T_+ - T_-] \to \Omega$ defined by $w^*(t) = u^*(t)$ for $t \in (T_-, T_+)$, where u^* is given by Theorem 1.1, and

$$\begin{aligned} & w^*(T_\pm) = x_\pm, \\ & w^*(T_+ + t) = u^*(T_+ - t), \quad t \in (0, T_+ - T_-]. \end{aligned}$$

□

The problem of existence of heteroclinic, homoclinic and periodic solutions of (1.2), in a context similar to the one considered here, was already discussed in [2] where $\partial\Omega$ is allowed to include continua of critical points. Our result concerning periodic solutions extends a corresponding result in [2] where existence was established under the assumption that $P = \emptyset$.

The following result is a direct consequence of Theorem 1.2.

Corollary 1.6. *Theorem 1.2 implies that, if all the sets $\Gamma_1, \dots, \Gamma_N$ have positive diameter, there exists $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$ and a periodic solution $v^* : \mathbb{R} \to \Omega$ of (1.2) and (1.4) that satisfies*

$$v^*(-t) = -v^*(t), \quad t \in \mathbb{R}.$$

This solution has period $T = 4T_+$, with T_+.

Proof. The solution v^* is the T-periodic extension of the map $w^* : [-2T_+, 2T_+] \to \Omega$ defined by $w^*(t) = u^*(t)$ for $t \in (0, T_+)$, where u^* is given by Theorem 1.2, and by

$$\begin{aligned} & w^*(t) = -w^*(-t), && t \in (-T_+, 0), \\ & w^*(0) = 0, \quad w^*(\pm T_+) = \pm x_+, \\ & w^*(T_+ + t) = w^*(T_+ - t), && t \in (0, T_+], \\ & w^*(-T_+ + t) = w^*(-T_+ - t), && t \in [-T_+, 0). \end{aligned}$$

In particular the solution oscillates between x_+ and $-x_+$ and this is true also when $\partial\Omega$ is connected ($N = 1$). □

2 Proof of Theorems 1.1 and 1.2

We recall a classical result.

Lemma 2.1. *Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded and non-negative potential, $I = (a, b)$ a bounded interval. Define the Jacobi functional*

$$\mathcal{J}_G(q, I) = \sqrt{2} \int_I \sqrt{G(q(t))} |\dot{q}(t)| dt$$

and the action functional

$$\mathcal{A}_G(q, I) = \int_I \Big(\frac{1}{2} |\dot{q}(t)|^2 + G(q(t)) \Big) dt.$$

Then

(i)

$$\mathcal{J}_G(q, I) \leq \mathcal{A}_G(q, I), \quad q \in W^{1,2}(I; \mathbb{R}^n)$$

with equality sign if and only if

$$\frac{1}{2} |\dot{q}(t)|^2 - G(q(t)) = 0, \ t \in I.$$

(ii)

$$\min_{q \in \mathcal{Q}} \mathcal{J}_G(q, I) = \min_{q \in \mathcal{Q}} \mathcal{A}_G(q, I),$$

where

$$\mathcal{Q} = \{q \in W^{1,2}(I; \mathbb{R}^n) : q(a) = q_a, q(b) = q_b\}.$$

When $G = U$ we shall simply write $\mathcal{J}, \mathcal{A}$ for $\mathcal{J}_U, \mathcal{A}_U$.

We now start the proof of Theorem 1.1. Choose $\Gamma_- \in \{\Gamma_1, \ldots, \Gamma_N\}$ and set

$$d = \min\{|x - y| : x \in \Gamma_-, y \in \partial\Omega \setminus \Gamma_-\}.$$

For small $\delta \in (0, d)$ let $O_\delta = \{x \in \Omega : d(x, \Gamma_-) < \delta\}$ and let $U_0 = \frac{1}{2} \min_{x \in \partial O_\delta \cap \Omega} U(x)$. We note that $U_0 > 0$ and define the admissible set

$$\begin{aligned} \mathcal{U} = \big\{ & u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty, \\ & u((T_-^u, T_+^u)) \subset \Omega, \ U(u(0)) = U_0, \ u(T_-^u) \in \Gamma_-, \ u(T_+^u) \in \partial\Omega \setminus \Gamma_- \big\}. \end{aligned} \tag{2.1}$$

We determine the map u^* in Theorem 1.1 as the limit of a minimizing sequence $\{u_j\} \subset \mathcal{U}$ of the action functional

$$\mathcal{A}(u, (T_-^u, T_+^u)) = \int_{T_-^u}^{T_+^u} \Big(\frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \Big) dt,$$

Note that in the definition of $\mathcal{U}$ the times T_-^u and T_+^u are not fixed but, in general, change with u. Note also that the condition $U(u(0)) = U_0$ in (2.1) is a normalization which can always be imposed by a translation of time and has the scope of eliminating the loss of compactness due to translation invariance. Let $\bar{x}_- \in \Gamma_-$ and $\bar{x}_+ \in \partial\Omega \setminus \Gamma_-$ be such that $|\bar{x}_+ - \bar{x}_-| = d$ and set

$$\tilde{u}(t) = (1 - (t + \tau))\bar{x}_- + (t + \tau)\bar{x}_+, \ \ t \in [-\tau, 1 - \tau],$$

where $\tau \in (0, 1)$ is chosen so that $U(\tilde{u}(0)) = U_0$. Then $\tilde{u} \in \mathcal{U}$, $T_-^{\tilde{u}} = -\tau$, $T_+^{\tilde{u}} = 1 - \tau$ and

$$\mathcal{A}(\tilde{u}, (-\tau, 1 - \tau)) = a < +\infty.$$

Next we show that there are constants $M > 0$ and $T_0 > 0$ such that each $u \in \mathcal{U}$ with

$$\mathcal{A}(u, (T_-^u, T_+^u)) \leq a, \tag{2.2}$$

satisfies

$$\begin{aligned} &\|u\|_{L^\infty((T_-^u, T_+^u);\mathbb{R}^n)} \leq M, \\ &T_-^u \leq -T_0 < T_0 \leq T_+^u. \end{aligned} \tag{2.3}$$

The L^∞ bound on u follows from **H** and from Lemma 2.1, in fact, if Ω is unbounded, $|u(\bar{t})| = M$ for some $\bar{t} \in (T_-^u, T_+^u)$ implies

$$a \geq \mathcal{A}(u, (T_-^u, \bar{t})) \geq \int_{T_-^u}^{\bar{t}} \sqrt{2U(u(t))}|\dot{u}(t)|dt \geq \sqrt{2} \int_{r_0}^{M} \sigma(s)ds.$$

The existence of T_0 follows from

$$\frac{d_1^2}{|T_-^u|} \leq \int_{T_-^u}^{0} |\dot{u}(t)|^2 dt \leq 2a, \qquad \frac{d_1^2}{T_+^u} \leq \int_{0}^{T_+^u} |\dot{u}(t)|^2 dt \leq 2a,$$

where $d_1 = d(\partial\Omega, \{x : U(x) > U_0\})$.

Let $\{u_j\} \subset \mathcal{U}$ be a minimizing sequence

$$\lim_{j\to+\infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) = \inf_{u\in\mathcal{U}} \mathcal{A}(u, (T_-^u, T_+^u)) := a_0 \leq a. \tag{2.4}$$

We can assume that each u_j satisfies (2.2) and (2.3). By considering a subsequence, that we still denote by $\{u_j\}$, we can also assume that there exist T_-^∞, T_+^∞ with $-\infty \leq T_-^\infty \leq -T_0 < T_0 \leq T_+^\infty \leq +\infty$ and a continuous map $u^* : (T_-^\infty, T_+^\infty) \to \mathbb{R}^n$ such that

$$\begin{aligned} &\lim_{j\to+\infty} T_\pm^{u_j} = T_\pm^\infty, \\ &\lim_{j\to+\infty} u_j(t) = u^*(t), \quad t \in (T_-^\infty, T_+^\infty), \end{aligned} \tag{2.5}$$

and in the last limit the convergence is uniform on bounded intervals. This follows from (2.3) which implies that the sequence $\{u_j\}$ is equi-bounded and from (2.2) which implies

$$|u_j(t_1) - u_j(t_2)| \leq \left| \int_{t_1}^{t_2} |\dot{u}_j(t)|dt \right| \leq \sqrt{a}|t_1 - t_2|^{\frac{1}{2}}, \tag{2.6}$$

so that the sequence is also equi-continuous.

By passing to a further subsequence we can also assume that $u_j \rightharpoonup u^*$ in $W^{1,2}((T_1, T_2); \mathbb{R}^n)$ for each T_1, T_2 with $T_-^\infty < T_1 < T_2 < T_+^\infty$. This follows from (2.2), which implies

$$\frac{1}{2} \int_{T_-^{u_j}}^{T_+^{u_j}} |\dot{u}_j|^2 dt \leq \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) \leq a,$$

and from the fact that each map u_j satisfies (2.3) and therefore is bounded in $L^2((T_-^{u_j}, T_+^{u_j}); \mathbb{R}^n)$. We also have

$$\mathcal{A}(u^*, (T_-^\infty, T_+^\infty)) \leq a_0. \tag{2.7}$$

Indeed, from the lower semicontinuity of the norm, for each T_1, T_2 with $T_-^\infty < T_1 < T_2 < T_+^\infty$ we have

$$\int_{T_1}^{T_2} |\dot{u}^*|^2 dt \leq \liminf_{j\to+\infty} \int_{T_1}^{T_2} |\dot{u}_j|^2 dt.$$

This and the fact that u_j converges to u^* uniformly in $[T_1, T_2]$ imply

$$\mathcal{A}(u^*, (T_1, T_2)) \leq \liminf_{j \to +\infty} \mathcal{A}(u_j, (T_1, T_2)) \leq \liminf_{j \to +\infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) = a_0.$$

Since this is valid for each $T_-^\infty < T_1 < T_2 < T_+^\infty$ the claim (2.7) follows.

Lemma 2.2. *Define* $T_-^\infty \leq T_- \leq -T_0 < T_0 \leq T_+ \leq T_+^\infty$ *by setting*

$$\begin{aligned} T_- &= \inf\{t \in (T_-^\infty, 0] : u^*((t, 0]) \subset \Omega\} \\ T_+ &= \sup\{t \in (0, T_+^\infty) : u^*([0, t)) \subset \Omega\}. \end{aligned}$$

Then

(i)

$$\mathcal{A}(u^*, (T_-, T_+)) = a_0. \tag{2.8}$$

(ii) $T_+ < +\infty$ *implies* $\lim_{t \to T_+} u^*(t) = x_+$ *for some* $x_+ \in \Gamma_+$ *and* $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\} \setminus \{\Gamma_-\}$.

(iii) $T_+ = +\infty$ *implies*

$$\lim_{t \to +\infty} d(u^*(t), \Gamma_+) = 0, \tag{2.9}$$

for some $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\} \setminus \{\Gamma_-\}$.

Corresponding statements apply to T_-.

Proof. We first prove (*ii*), (*iii*). If $T_+ < +\infty$ the existence of $\lim_{t \to T_+} u^*(t)$ follows from (2.6) which implies that u^* is a $C^{0,\frac{1}{2}}$ map. The limit x_+ belongs to $\partial\Omega$ and therefore to Γ_+ for some $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\}$. Indeed, $x_+ \notin \partial\Omega$ would imply the existence of $\tau > 0$ such that, for j large enough,

$$d(u_j([T_+, T_+ + \tau]), \partial\Omega) \geq \frac{1}{2} d(x_+, \partial\Omega),$$

in contradiction with the definition of T_+. If $T_+ = +\infty$ and (iii) does not hold there is $\delta > 0$ and a diverging sequence $\{t_j\}$ such that

$$d(u^*(t_j), \partial\Omega) \geq \delta.$$

Set $U_m = \min_{d(x, \partial\Omega) = \delta} U(x) > 0$. From the uniform continuity of U in $\{|x| \leq M\}$ (M as in (2.3)) it follows that there is $l > 0$ such that

$$|U(x_1) - U(x_2)| \leq \frac{1}{2} U_m, \quad \text{for } |x_1 - x_2| \leq l, \; x_1, x_2 \in \{|x| \leq M\}.$$

This and $u^* \in C^{0,\frac{1}{2}}$ imply

$$U(u^*(t)) \geq \frac{1}{2} U_m, \;\; t \in I_j = \Big(t_j - \frac{l^2}{a}, t_j + \frac{l^2}{a}\Big),$$

and, by passing to a subsequence, we can assume that the intervals I_j are disjoint. Therefore for each $T > 0$ we have

$$\sum_{t_j \leq T} \frac{l^2 U_m}{a} \leq \int_0^T U(u^*(t)) dt \leq a_0,$$

which is impossible for T large. This establishes (2.9) for some $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\}$. It remains to show that $\Gamma_+ \neq \Gamma_-$. This is a consequence of the minimizing character of $\{u_j\}$. Indeed, $\Gamma_+ = \Gamma_-$ would imply the existence of a constant $c > 0$ such that $\lim_{j \to \infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) \geq a_0 + c$.

Now we prove (i). $T_+ - T_- < +\infty$, implies that u^* is an element of $\mathcal{U}$ with $T_\pm^{u^*} = T_\pm$. It follows that $\mathcal{A}(u^*, (T_-, T_+)) \geq a_0$, which together with (2.7) imply (2.8). Assume now $T_+ - T_- = +\infty$. If $T_+ = +\infty$, (2.9) implies that, given a small number $\epsilon > 0$, there are t_ϵ and $\bar{x}_\epsilon \in \partial\Omega$ such that $|u^*(t_\epsilon) - \bar{x}_\epsilon| = \epsilon$ and the segment joining $u^*(t_\epsilon)$ to $\bar{x}_\epsilon$ belongs to $\overline{\Omega}$. Set

$$v_\epsilon(t) = (1 - (t - t_\epsilon))u^*(t_\epsilon) + (t - t_\epsilon)\bar{x}_\epsilon, \;\; t \in (t_\epsilon, t_\epsilon + 1].$$

From the uniform continuity of U there is $\eta_\epsilon > 0$, $\lim_{\epsilon\to 0} \eta_\epsilon = 0$, such that $U(v_\epsilon(t)) \leq \eta_\epsilon$, for $t \in [t_\epsilon, t_\epsilon + 1]$. Therefore we have

$$\mathcal{A}(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq \frac{1}{2}\epsilon^2 + \eta_\epsilon.$$

If $T_- > -\infty$ the map $u_\epsilon = \mathbb{1}_{[T_-, t_\epsilon]}u^* + \mathbb{1}_{(t_\epsilon, t_\epsilon+1]}v_\epsilon$ belongs to $\mathcal{U}$ and it results

$$a_0 \leq \mathcal{A}(u_\epsilon, (T_-, t_\epsilon + 1)) = \mathcal{A}(u^*, (T_-, t_\epsilon)) + \mathcal{A}(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq \mathcal{A}(u^*, (T_-, T_+)) + \frac{1}{2}\epsilon^2 + \eta_\epsilon.$$

Since this is valid for all small $\epsilon > 0$ we get

$$a_0 \leq \mathcal{A}(u^*, (T_-, T_+)),$$

that together with (2.7) establishes (2.8) if $T_- > -\infty$ and $T_+ = +\infty$. The discussion of the other cases where $T_+ - T_- = +\infty$ is similar. □

We observe that there are cases with $T_+ < T_+^\infty$ and/or $T_- > T_-^\infty$, see Remark 2.

Lemma 2.3. *The map u^* satisfies (1.2) and (1.4) in (T_-, T_+).*

Proof. 1. We first show that for each T_1, T_2 with $T_- < T_1 < T_2 < T_+$ we have

$$\mathcal{A}(u^*, (T_1, T_2)) = \inf_{v\in\mathcal{V}} \mathcal{A}(v, (T_1, T_2)), \tag{2.10}$$

where

$$\mathcal{V} = \{v \in W^{1,2}((T_1, T_2); \mathbb{R}^n) : v(T_i) = u^*(T_i), i = 1, 2; \; v([T_1, T_2]) \subset \Omega\}.$$

Suppose instead that there are $\eta > 0$ and $v \in \mathcal{V}$ such that

$$\mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta.$$

Set $w_j : (T_-^{u_j}, T_+^{u_j}) \to \Omega$ defined by

$$w_j(t) = \begin{cases} u_j(t), \;\; t \in (T_-^{u_j}, T_1] \cup [T_2, T_+^{u_j}), \\ v(t) + \dfrac{T_2 - t}{T_2 - T_1}\delta_{1j} + \dfrac{t - T_1}{T_2 - T_1}\delta_{2j}, \;\; t \in (T_1, T_2), \end{cases}$$

where $\delta_{ij} = u_j(T_i) - u^*(T_i)$, $i = 1, 2$, with u_j as in (2.4). Define $v_j : [T_-^{v_j}, T_+^{v_j}] \to \mathbb{R}^n$ by

$$v_j(t) = w_j(t - \tau_j),$$

where τ_j is such that $U(v_j(0)) = U_0$, as in (2.1). Note that

$$\mathcal{A}(v_j, (T_-^{v_j}, T_+^{v_j})) = \mathcal{A}(w_j, (T_-^{u_j}, T_+^{u_j})). \tag{2.11}$$

From (2.5) we have $\lim_{j\to\infty} \delta_{ij} = 0, i = 1, 2$, so that

$$\lim_{j\to+\infty} \mathcal{A}(w_j, (T_1, T_2)) = \mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta \leq \liminf_{j\to+\infty} \mathcal{A}(u_j, (T_1, T_2)) - \eta.$$

Therefore we have

$$\begin{aligned}&\liminf_{j\to+\infty}\mathcal{A}(w_j,(T_-^{u_j},T_+^{u_j}))=\lim_{j\to+\infty}\mathcal{A}(w_j,(T_1,T_2))+\liminf_{j\to+\infty}\mathcal{A}(u_j,(T_+^{u_j},T_1)\cup(T_2,T_+^{u_j}))\\&\le\liminf_{j\to+\infty}\mathcal{A}(u_j,(T_1,T_2))-\eta+\liminf_{j\to+\infty}\mathcal{A}(u_j,(T_+^{u_j},T_1)\cup(T_2,T_+^{u_j}))\le a_0-\eta,\end{aligned}$$

that, given (2.11), is in contradiction with the minimizing character of the sequence $\{u_j\}$.
The fact that u^* satisfies (1.2) follows from (2.10) and regularity theory, see [5]. To show that u^* satisfies (1.4) we distinguish the case $T_+-T_-<+\infty$ from the case $T_+-T_-=+\infty$.
2. $T_+-T_-<+\infty$. Given t_0,t_1 with $T_-<t_0<t_1<T_+$, let $\phi:[t_0,t_1+\tau]\to[t_0,t_1]$ be linear, with $|\tau|$ small, and let $\psi:[t_0,t_1]\to[t_0,t_1+\tau]$ be the inverse of ϕ. Define $u_\tau:[T_-,T_++\tau]\to\mathbb{R}^n$ by setting

$$u_\tau(t)=\begin{cases}u^*(t), & t\in[T_-,t_0],\\ u^*(\phi(t)), & t\in[t_0,t_1+\tau],\\ u^*(t-\tau), & t\in(t_1+\tau,T_++\tau)]\end{cases}\tag{2.12}$$

Note that $u_\tau\in\mathcal{U}$ with $T_-^{u_\tau}=T_-$ and $T_+^{u_\tau}=T_++\tau$. Since u^* is a minimizer we have

$$\frac{d}{d\tau}\mathcal{A}(u_\tau,(T_-^{u_\tau},T_+^{u_\tau}))|_{\tau=0}=0.\tag{2.13}$$

From (2.12), using also the change of variables $t=\psi(s)$, it follows

$$\begin{aligned}&\mathcal{A}(u_\tau,(T_-^{u_\tau},T_+^{u_\tau}))-\mathcal{A}(u^*,(T_-,T_+))\\&=\int_{t_0}^{t_1+\tau}\Big(\frac{\dot\phi^2(t)}{2}|\dot u^*(\phi(t))|^2+U(u^*(\phi(t)))\Big)dt-\int_{t_0}^{t_1}\Big(\frac{1}{2}|\dot u^*(t)|^2+U(u^*(t))\Big)dt\\&=\int_{t_0}^{t_1}\Big(\frac{1-\dot\psi(t)}{2\dot\psi(t)}|\dot u^*(t)|^2+(\dot\psi(t)-1)U(u^*(t))\Big)dt\\&=\int_{t_0}^{t_1}\Big(\frac{-\frac{\tau}{t_1-t_0}}{2(1+\frac{\tau}{t_1-t_0})}|\dot u^*(t)|^2+\frac{\tau}{t_1-t_0}U(u^*(t))\Big)dt\\&=-\frac{\tau}{t_1-t_0}\int_{t_0}^{t_1}\Big(\frac{|\dot u^*(t)|^2}{2(1+\frac{\tau}{t_1-t_0})}-U(u^*(t))\Big)dt.\end{aligned}$$

This and (2.13) imply

$$\int_{t_0}^{t_1}\Big(\frac{1}{2}|\dot u^*(t)|^2-U(u^*(t))\Big)dt=0.\tag{2.14}$$

Since this holds for all t_0,t_1, with $T_-<t_0<t_1<T_+$, then (1.4) follows.
3. $T_+-T_-=+\infty$. We only consider the case $T_+=+\infty$. The discussion of the other cases is similar. Let $T\in(T_-,+\infty)$, let $T_-<t_0<t_1<T$ and let $\phi:[t_0,T]\to[t_0,T]$ be linear in the intervals $[t_0,t_1+\tau]$, $[t_1+\tau,T]$, with $|\tau|$ small, and such that $\phi([t_0,t_1+\tau])=[t_0,t_1]$. Define $u_\tau:(T_-,+\infty)\to\mathbb{R}^n$ by setting

$$u_\tau(t)=\begin{cases}u^*(t), & t\in(T_-,t_0]\cup[T,+\infty)\\ u^*(\phi(t)), & t\in[t_0,T].\end{cases}$$

We have

$$\begin{aligned}&\mathcal{A}(u_\tau,(T_-,T))-\mathcal{A}(u^*,(T_-,T))\\&=\int_{t_0}^{t_1}\Big(\frac{-\frac{\tau}{t_1-t_0}}{2(1+\frac{\tau}{t_1-t_0})}|\dot u^*(t)|^2+\frac{\tau}{t_1-t_0}U(u^*(t))\Big)dt+\int_{t_1}^{T}\Big(\frac{\frac{\tau}{T-t_1}}{2(1+\frac{\tau}{T-t_1})}|\dot u^*(t)|^2-\frac{\tau}{T-t_1}U(u^*(t))\Big)dt.\end{aligned}$$

Since u^* restricted to the interval $[t_0, T]$ is a minimizer of (2.10), by differentiating with respect to τ and setting $\tau = 0$ we obtain

$$-\frac{1}{t_1 - t_0}\int_{t_0}^{t_1}\Big(\frac{1}{2}|\dot{u}^*(t)|^2 - U(u^*(t))\Big)dt + \frac{1}{T - t_1}\int_{t_1}^{T}\Big(\frac{1}{2}|\dot{u}^*(t)|^2 - U(u^*(t))\Big)dt = 0.$$

From (2.7) it follows that the second term in this expression converges to zero when $T \to +\infty$. Therefore, after taking the limit for $T \to +\infty$, we get back to (2.14) and, as before, we conclude that (1.4) holds. □

Lemma 2.4. *Assume that* $\lim_{t\to T_+} u^*(t) = p \in P$. *Then*

$$T_+ = +\infty.$$

Proof. Since U is of class C^2 and p is a critical point of U there are constants $c > 0$ and $\rho > 0$ such that

$$U(x) \le c|x - p|^2, \ \ x \in B_\rho(p) \cap \Omega.$$

Fix t_ρ so that $u^*(t) \in B_\rho(p) \cap \Omega$ for $t \ge t_\rho$. Then $T_+ = +\infty$ follows from (1.4) and

$$\frac{d}{dt}|u^* - p| \ge -|\dot{u}^*| = -\sqrt{2U(u^*)} \ge -\sqrt{2c}|u^* - p|, \quad t \ge t_\rho.$$

□

We now show that if Γ_+ has positive diameter then $T_+ < +\infty$. To prove this we first show that $T_+ = +\infty$ implies $u^*(t) \to p \in P$ as $t \to +\infty$, then we conclude that this is in contrast with (2.8).

Lemma 2.5. *If* $T_+ = +\infty$*, then there is* $p \in P$ *such that*

$$\lim_{t\to+\infty} u^*(t) = p. \tag{2.15}$$

An analogous statement applies to T_-.

Proof. If $\Gamma_+ = \{p\}$ for some $p \in P$, then (2.15) follows by (2.9). Therefore we assume that Γ_+ has positive diameter. The idea of the proof is to show that if $u^*(t)$ gets too close to $\partial\Gamma_+ \setminus P$ it is forced to end up on $\Gamma_+ \setminus P$ in a finite time in contradiction with $T^* = +\infty$.

If (2.15) does not hold there is $q > 0$ and a sequence $\{\tau_j\}$, with $\lim_{j\to\infty}\tau_j = +\infty$, such that $d(u^*(\tau_j), P) \ge q$, for all $j \in \mathbb{N}$. Since, by (2.3) u^* is bounded, using also (2.9), we can assume that

$$\lim_{j\to+\infty} u^*(\tau_j) = \bar{x}, \ \text{ for some } \ \bar{x} \in \Gamma_+ \setminus \cup_{p\in P} B_q(p). \tag{2.16}$$

The smoothness of U implies that there are positive constants $\bar{r}$, r, c and C such that

(i) the orthogonal projection on $\pi : B_{\bar{r}}(\bar{x}) \to \partial\Omega$ is well defined and $\pi(B_{\bar{r}}(\bar{x})) \subset \partial\Omega \setminus P$;

(ii) we have

$$B_r(x_0) \subset B_{\bar{r}}(\bar{x}), \ \text{ for all } x_0 \in \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x});$$

(iii) if $(\xi, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are local coordinates with respect to a basis $\{e_1, \dots, e_n\}$, $e_j = e_j(x_0)$, with $e_n(x_0)$ the unit interior normal to $\partial\Omega$ at $x_0 \in \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x})$ it results

$$\frac{1}{2}cs \le U(x(x_0, (\xi, s))) \le 2cs, \ \ |\xi|^2 + s^2 \le r^2, \ s \ge h(x_0, \xi), \tag{2.17}$$

where

$$x = x(x_0, (\xi, s)) = x_0 + \sum_{j=1}^{n}\xi_j e_j(x_0) + s e_n(x_0),$$

and $h : \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x}) \times \{|\xi| \le r\} \to \mathbb{R}$, $|h(x_0, \xi)| \le C|\xi|^2$, for $|\xi| \le r$, is a local representation of $\partial\Omega$ in a neighborhood of x_0, that is $U(x(x_0, (\xi, h(x_0, \xi)))) = 0$ for $|\xi| \le r$.

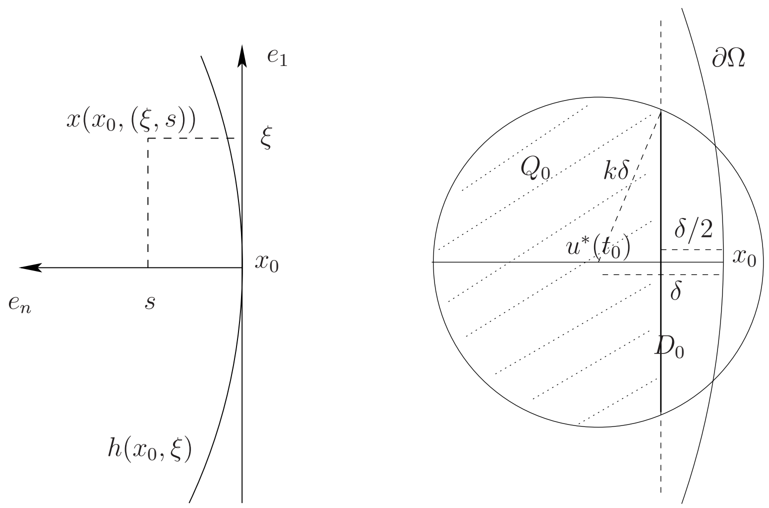

Figure 1: The coordinates (ξ, s) and the domain Q_0 in Lemma 2.5.

Fix a value j_0 of j and set $t_0 = \tau_{j_0}$. If j_0 is sufficiently large, setting $t_0 = \tau_{j_0}$ we have that $x_0 = \pi(u^*(t_0))$ is well defined. Moreover $x_0 \in \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x})$ and

$$u^*(t_0) = x_0 + \delta e_n(x_0), \quad \delta = |u^*(t_0) - x_0|.$$

For $k = \frac{8}{3}\sqrt{2}$ let Q_0 be the set

$$Q_0 = \{x(x_0, (\xi, s)) : |\xi|^2 + (s - \delta)^2 < k^2\delta^2, \ s > \delta/2\}.$$

Since $\delta \to 0$ as $j_0 \to +\infty$ we can assume that $\delta > 0$ is so small ($\delta < \min\{\frac{1}{2Ck^2}, \frac{r}{1+k}\}$ suffices) that $\overline{Q}_0 \subset \Omega \cap B_r(x_0)$.

Claim 1. $u^*(t)$ leaves $\overline{Q}_0$ through the disc $D_0 = \partial Q_0 \setminus \partial B_{k\delta}(u^*(t_0))$.

From (2.4) we have $a_0 \le \mathcal{A}(v, (T_-, T_+^v))$ for each $W^{1,2}$ map $v : (T_-, T_+^v] \to \mathbb{R}^n$ that coincides with u^* for $t \le t_0$, and satisfies $v((t_0, T_+^v)) \subset \Omega$, $v(T_+^v) \in \partial\Omega$ and (1.4). Therefore if we set

$$w(s) = x_0 + s e_n(x_0),$$

$s \in [0, \delta]$, we have

$$a_0 \le \mathcal{A}(u^*, (T_-, t_0)) + \mathcal{J}(w, (0, \delta)). \tag{2.18}$$

On the other hand, if $u^*(t_0') \in \partial Q_0(x_0) \cap \partial B_{k\delta}(u^*(t_0))$, where

$$t_0' = \sup\{t > t_0 : u^*([t_0, t)) \subset \overline{Q}_0 \setminus \partial B_{k\delta}(u^*(t_0))\},$$

from (2.7) it follows

$$\mathcal{A}(u^*, (T_-, t_0)) + \mathcal{J}(u^*, (t_0, t_0')) \le a_0. \tag{2.19}$$

Using (2.17) we obtain

$$\mathcal{J}(w, (0, \delta)) \le \frac{4}{3} c^{\frac{1}{2}} \delta^{\frac{3}{2}}, \tag{2.20}$$

and, since

$$c\frac{\delta}{4} \le U(x(x_0, (\xi, s))), \quad (\xi, s) \in \overline{Q}_0(x_0),$$

we also have, with k defined above,

$$\frac{8}{3}c^{\frac{1}{2}}\delta^{\frac{3}{2}} = \frac{k}{\sqrt{2}}c^{\frac{1}{2}}\delta^{\frac{3}{2}} \leq \frac{c^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\sqrt{2}}\int_{t_0}^{t_0'}|\dot{u}^*(t)|dt \leq \sqrt{2}\int_{t_0}^{t_0'}\sqrt{U(u^*(t))}|\dot{u}^*(t)|dt. \tag{2.21}$$

From (2.20) and (2.21) it follows

$$\mathcal{J}(w,(0,\delta)) \leq \frac{1}{2}\mathcal{J}(u^*,(t_0,t_0')),$$

and therefore (2.18) and (2.19) imply the absurd inequality $a_0 < a_0$. This contradiction proves the claim.

From Claim 1 it follows that there is $t_1 \in (t_0,+\infty)$ with the following properties:

$$\begin{aligned}&u^*([t_0,t_1)) \subset Q_0(x_0),\\&u(t_1) \in D_0.\end{aligned}$$

Set $x_{0,1} = \pi(u^*(t_1))$ and $\delta_1 = |u^*(t_1) - x_{0,1}|$. Since $h(x_0,0) = h_\xi(x_0,0) = 0$ and the radius $\rho_\delta = (k^2-\frac{1}{4})^{\frac{1}{2}}\delta$ of D_0 is proportional to δ, we can assume that δ is so small that the ratio $\frac{2\delta_1}{\delta}$ and $\frac{|x_{0,1}-x_0|}{|u^*(t_1)-x(x_0,(0,\frac{\delta}{2}))|}$ are near 1 so that we have

$$\begin{aligned}&\delta_1 \leq \rho\delta, \ \text{ for some } \ \rho<1,\\&|x_{0,1}-x_0| \leq k\delta.\end{aligned}$$

We also have

$$t_1 - t_0 \leq k'\delta^{\frac{1}{2}}, \ \ k' = \frac{8k}{c^{\frac{1}{2}}}.$$

This follows from

$$\begin{aligned}&(t_1-t_0)\frac{c}{4}\delta \leq \mathcal{A}(u^*,(t_0,t_1)) = \mathcal{J}(u^*,(t_0,t_1))\\&= \sqrt{2}\int_{t_0}^{t_1}\sqrt{U(u^*(t))}|\dot{u^*}(t)|dt \leq 2\sqrt{c\delta}|u^*(t_1)-u^*(t_0)| \leq 2c^{\frac{1}{2}}k\delta^{\frac{3}{2}}.\end{aligned}$$

where we used (2.17) to estimate $\mathcal{J}$ on the segment joining $u^*(t_0)$ with $u^*(t_1)$.
We have $u^*(t_1) = x_{0,1} + \delta_1 e_n(x_{0,1})$ and we can apply Claim 1 to deduce that there exists $t_2 > t_1$ such that

$$\begin{aligned}&u^*([t_1,t_2)) \subset Q_1(x_{0,1}),\\&u^*(t_2) \in D_1,\end{aligned}$$

where Q_1 and D_1 are defined as Q_0 and D_0 with δ_1 and $x(x_{0,1},(\xi,s))$ instead of δ and $x(x_0,(\xi,s))$. Therefore an induction argument yields sequences $\{t_j\}$, $\{x_{0,j}\}$, $\{\delta_j\}$ and $\{Q_j(x_{0,j})\}$ such that

$$\begin{aligned}&u^*([t_j,t_{j+1})) \subset Q_j(x_{0,j}), \quad x_{0,j} = \pi(u^*(t_j)),\\&\delta_{j+1} \leq \rho\delta_j \leq \rho^{j+1}\delta,\\&|x_{0,j+1}-x_{0,j}| \leq k\delta_j \leq k\rho^j\delta,\\&(t_{j+1}-t_j) \leq k'\delta_j^{1/2} \leq k'\rho^{j/2}\delta^{1/2},\\&u^*(t_j) = x_{0,j} + \delta_j e_n(x_{0,j}) \in D_j.\end{aligned} \tag{2.22}$$

We can also assume that $Q_j(x_{0,j}) \subset \Omega \cap B_r(x_0)$, for all $j \in \mathbb{N}$. This follows from $|u^*(t_{j+1}) - u^*(t_j)| \leq k\delta_j \leq k\rho^j\delta$.

From (2.22) we obtain that there exists T with $t_0 < T \leq \frac{k'\delta^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}$ such that

$$u^*(T) = \lim_{t \to T} u^*(t) = \lim_{j \to +\infty} x_{0,j} \in \partial\Omega \setminus P,$$
$$|u^*(T) - x_0| \leq \frac{k\delta}{1-\rho}.$$

This contradicts the existence of the sequence $\{\tau_j\}$, with $\lim_{j\to\infty} \tau_j = +\infty$, appearing in (2.16) and establishes (2.15). The proof of the lemma is complete. □

We continue by showing (2.15) contradicts (2.8).

Lemma 2.6. *Assume that* Γ_+ *has positive diameter. Then*

$$T_+ < +\infty.$$

An analogous statement applies to Γ_- *and* T_-.

Proof. From Lemma 2.5, if $T_+ = +\infty$ there exists $p \in P$ such that $\lim_{t\to+\infty} u^*(t) = p$. We use a local argument to show that this is impossible if Γ_+ has positive diameter. By a suitable change of variable we can assume that $p = 0$ and that, in a neighborhood of $0 \in \mathbb{R}^n$, U reads

$$U(u) = V(u) + W(u),$$

where V is the quadratic part of U:

$$V(u) = \frac{1}{2}\Big(-\sum_{i=1}^{m} \lambda_i^2 u_i^2 + \sum_{i=m+1}^{n} \lambda_i^2 u_i^2\Big), \qquad \lambda_i > 0 \tag{2.23}$$

and W satisfies,

$$|W(u)| \leq C|u|^3, \quad |W_x(u)| \leq C|u|^2, \quad |W_{xx}(u)| \leq C|u|. \tag{2.24}$$

Consider the Hamiltonian system with

$$H(p,q) = \frac{1}{2}|p|^2 - U(q), \quad p \in \mathbb{R}^n,\ q \in \Omega \subset \mathbb{R}^n.$$

For this system the origin of $\mathbb{R}^{2n}$ is an equilibrium point that corresponds to the critical point $p = 0$ of U. Set $D = \mathrm{diag}(-\lambda_1^2, \dots, -\lambda_m^2, \lambda_{m+1}^2, \dots, \lambda_n^2)$. The eigenvalues of the symplectic matrix

$$\begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix}$$

are

$$\begin{aligned} -\lambda_i,&\ \ i = m+1, \dots, n \\ \lambda_i,&\ \ i = m+1, \dots, n \\ \pm i\lambda_i,&\ \ i = 1, \dots, m. \end{aligned}$$

Let $(e_1, 0), \dots, (e_n, 0)$, $(0, e_1), \dots, (0, e_n)$ be the basis of $\mathbb{R}^{2n}$ defined by $e_j = (\delta_{j1}, \dots, \delta_{jn})$, where δ_{ji} is Kronecker's delta. The stable S^s, unstable S^u and center S^c subspaces invariant under the flow of the linearized Hamiltonian system at $0 \in \mathbb{R}^{2n}$ are

$$\begin{aligned} S^s &= \mathrm{span}\{(-\lambda_j e_j, e_j)\}_{j=m+1}^n, \\ S^u &= \mathrm{span}\{(\lambda_j e_j, e_j)\}_{j=m+1}^n, \\ S^c &= \mathrm{span}\{(e_j, 0), (0, e_j)\}_{j=1}^m. \end{aligned}$$

From (2.15) and (1.4) we have

$$\lim_{t\to+\infty}(\dot{u}^*(t),u^*(t))=0\in\mathbb{R}^{2n}.$$

Let W^s and W^u be the local stable and unstable manifold and let W^c be a local center manifold at $0\in\mathbb{R}^{2n}$. From the center manifold theorem [4], [10], there is a constant $\lambda_0>0$ such that, for each solution $(p(t),q(t))$ that remains in a neighborhood of $0\in\mathbb{R}^{2n}$ for positive time, there is a solution $(p^c(t),q^c(t))\in W^c$ that satisfies

$$|(p(t),q(t))-(p^c(t),q^c(t))|=\mathrm{O}(e^{-\lambda_0 t}). \tag{2.25}$$

Since W^c is tangent to S^c at $0\in\mathbb{R}^{2n}$, the projection W_0^c on the configuration space is tangent to $S_0^c=\mathrm{span}\{e_j\}_{j=1}^m$, which is the projection of S^c on the configuration space. Therefore, if $(p^c,q^c)\not\equiv 0$, given $\gamma>0$, by (2.25) there is t_γ such that $d(q(t),S_0^c)\leq\gamma|q(t)|$, for $t\geq t_\gamma$. For γ small, this implies that $q(t)\notin\Omega$ for $t\geq t_\gamma$. It follows that $(p^c,q^c)\equiv 0$ and from (2.25) $(p(t),q(t))$ converges to zero exponentially. This is possible only if $(p(t),q(t))\in W^s$ and, in turn, only if $q(t)\in W_0^s$, the projection of W^s on the configuration space. This argument leads to the conclusion that the trajectory of u^* in a neighborhood of 0 is of the form

$$u^*(t(s))=\mathfrak{u}^*(s)=s\eta+z(s), \tag{2.26}$$

where

$$\eta=\sum_{i=m+1}^{n}\eta_i e_i$$

is a unit vector[1], $s\in[0,s_0)$ for some $s_0>0$, and $z(s)$ satisfies

$$z(s)\cdot\eta=0,\quad |z(s)|\leq c|s|^2,\quad |z'(s)|\leq c|s| \tag{2.27}$$

for a positive constant c.

We are now in the position of constructing our local perturbation of u. We first discuss the case $U=V$, $z(s)=0$. We set

$$\bar{u}(s)=s\eta$$

and, in some interval $[1,s_1]$, construct a competing map $\bar{v}:[1,s_1]\to\mathbb{R}^n$,

$$\bar{v}=\bar{u}+ge_1,\quad g:[1,s_1]\to\mathbb{R},$$

with the following properties:

$$\begin{aligned}&V(\bar{v}(1))=0,\\&\bar{v}(s_1)=\bar{u}(s_1),\\&\mathcal{J}_V(\bar{v},[1,s_1])<\mathcal{J}_V(\bar{u},[0,s_1]).\end{aligned} \tag{2.28}$$

The basic observation is that, if we move from $\bar{u}$ in the direction of one of the eigenvectors $e_1,\dots,e_m$ corresponding to negative eigenvalues of the Hessian of V, the potential V decreases and therefore, for each $s_0\in(1,s_1)$ we can define the function g in the interval $[1,s_0]$ so that

$$\mathcal{J}_V(\bar{u}+ge_1,(1,s_0))=\mathcal{J}_V(\bar{u},(1,s_0)). \tag{2.29}$$

Indeed it suffices to impose that $g:(1,s_0]\to\mathbb{R}$ satisfies the condition

$$\sqrt{V(\bar{u}(s))}=\sqrt{1+g'^2(s)}\sqrt{V(\bar{u}(s)+g(s)e_1)},\quad s\in(1,s_0].$$

[1]Actually η coincides with one of the eigenvectors of $U''(0)$.

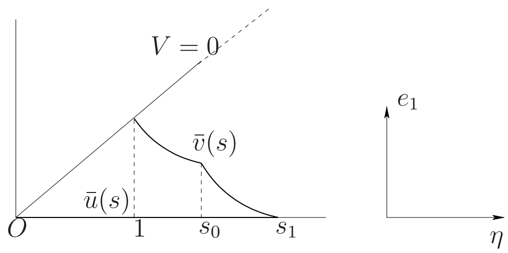

Figure 2: The maps $\bar{u}(s)$ and $\bar{v}(s)$.

According with this condition we take g as the solution of the problem

$$\begin{cases} g' = -\dfrac{\lambda_1 g}{\sqrt{s^2\lambda_\eta^2 - \lambda_1^2 g^2}} = -\dfrac{\frac{\lambda_1 g}{s\lambda_\eta}}{\sqrt{1 - \frac{\lambda_1^2 g^2}{s^2\lambda_\eta^2}}} \\ g(1) = \frac{\lambda_\eta}{\lambda_1} \end{cases}, \tag{2.30}$$

where we have used (2.23) and set

$$\lambda_\eta = \sqrt{\sum_{i=m+1}^{n} \lambda_i^2 \eta_i^2}.$$

Note that the initial condition in (2.30) implies $V(\bar{v}(1)) = 0$. The solution g of (2.30) is well defined in spite of the fact that the right hand side tends to $-\infty$ as $s \to 1$. Since g defined by (2.30) is positive for $s \in [1, +\infty)$, to satisfy the condition $\bar{v}(s_1) = \bar{u}(s_1)$, we give a suitable definition of g in the interval $[s_0, s_1]$ in order that $g(s_1) = 0$. Choose a number $\alpha \in (0, 1)$ and extend g with continuity to the interval $[s_0, s_1]$ by imposing that

$$\sqrt{V(\bar{u}(s))} = \alpha\sqrt{1 + g'^2(s)}\sqrt{V(\bar{u}(s) + g(s)e_1)}, \quad s \in (s_0, s_1]. \tag{2.31}$$

Therefore, in the interval $(s_0, s_1]$, we define g by

$$g' = -\frac{1}{\alpha}\sqrt{\frac{1 - \alpha^2 + \alpha^2\frac{\lambda_1^2 g^2}{s^2\lambda_\eta^2}}{1 - \frac{\lambda_1^2 g^2}{s^2\lambda_\eta^2}}} \le -\frac{\sqrt{1-\alpha^2}}{\alpha}. \tag{2.32}$$

Since (2.31) implies

$$\mathcal{J}_V(\bar{v}, [s_0, s_1]) = \frac{1}{\alpha}\mathcal{J}_V(\bar{u}, [s_0, s_1]),$$

from (2.29) we see that $\bar{v}$ satisfies also the requirement (2.28) above if we can choose $\alpha \in (0, 1)$ and $1 < s_0 < s_1$ in such a way that

$$\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1-\alpha}{\alpha}\mathcal{J}_V(\bar{u}, (s_0, s_1)).$$

Since (2.32) implies $s_1 < s_0 + \frac{\alpha g(s_0)}{\sqrt{1-\alpha^2}}$ a sufficient condition for this is

$$\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1-\alpha}{\alpha}\mathcal{J}_V\Big(\bar{u}, \Big(s_0, s_0 + \frac{\alpha g(s_0)}{\sqrt{1-\alpha^2}}\Big)\Big),$$

or equivalently

$$1>\frac{1-\alpha}{\alpha}\Big(\Big(s_0+\frac{\alpha g(s_0)}{\sqrt{1-\alpha^2}}\Big)^2-s_0^2\Big)=2s_0g(s_0)\sqrt{\frac{1-\alpha}{1+\alpha}}+\frac{\alpha g^2(s_0)}{1+\alpha}. \tag{2.33}$$

By a proper choice of s_0 and α the right hand side of (2.33) can be made as small as we like. For instance we can fix s_0 so that $g(s_0)\leq\frac{1}{4}$ and then choose α in such a way that $\frac{1}{2}s_0\sqrt{\frac{1-\alpha}{1+\alpha}}\leq\frac{1}{4}$ and conclude that (2.28) holds.

Next we use the function g to define a comparison map v that coincides with u^* outside an ϵ-neighborhood of 0 and show that the assumption that the trajectory of u^* ends up in some $p\in P$ must be rejected. For small $\epsilon>0$ we define

$$v(\epsilon s)=\epsilon s\eta+z(\epsilon s)+\epsilon g(s-\sigma)e_1,\quad s\in[1+\sigma,s_1+\sigma], \tag{2.34}$$

where $\sigma=\sigma(\epsilon)$ is determined by the condition

$$U(v(\epsilon(1+\sigma)))=0,$$

which, using (2.23), (2.24), (2.27) and $g(1)=\frac{\lambda_\eta}{\lambda_1}$, after dividing by ϵ^2, becomes

$$\frac{1}{2}\lambda_\eta^2((1+\sigma)^2-1)=\epsilon f(\sigma,\epsilon), \tag{2.35}$$

where $f(\sigma,\epsilon)$ is a smooth bounded function defined in a neighborhood of $(0,0)$. For small $\epsilon>0$, there is a unique solution $\sigma(\epsilon)=\mathrm{O}(\epsilon)$ of (2.35). Note also that (2.34) implies that

$$v(\epsilon(s_1+\sigma))=\mathfrak{u}^*(\epsilon(s_1+\sigma)).$$

We now conclude by showing that, for $\epsilon>0$ small, it results

$$\mathcal{J}_U(\mathfrak{u}^*(\epsilon\cdot),(0,s_1+\sigma))>\mathcal{J}_U(v(\epsilon\cdot),(1+\sigma,s_1+\sigma)). \tag{2.36}$$

From (2.26) and (2.34) we have

$$\lim_{\epsilon\to0^+}\epsilon^{-1}\Big|\frac{d}{ds}\mathfrak{u}^*(\epsilon s)\Big|=1,\quad\lim_{\epsilon\to0^+}\epsilon^{-1}\Big|\frac{d}{ds}v(\epsilon s)\Big|=\sqrt{1+g'^2(s)}, \tag{2.37}$$

and, using also (2.24) and $\sigma=\mathrm{O}(\epsilon)$,

$$\begin{aligned}&\lim_{\epsilon\to0^+}\epsilon^{-2}U(\mathfrak{u}^*(\epsilon s))=V(\bar{u}(s)),\quad s\in(0,s_1),\\&\lim_{\epsilon\to0^+}\epsilon^{-2}U(v(\epsilon s))=V(\bar{v}(s)),\quad s\in(1,s_1)\end{aligned} \tag{2.38}$$

uniformly in compact intervals.

The limits (2.37) and (2.38) imply

$$\begin{aligned}&\lim_{\epsilon\to0^+}\epsilon^{-2}\mathcal{J}_U(\mathfrak{u}^*(\epsilon\cdot),(0,s_1+\sigma))=\lim_{\epsilon\to0^+}\sqrt{2}\int_0^{s_1+\sigma}\sqrt{\epsilon^{-2}U(\mathfrak{u}^*(\epsilon s))}\epsilon^{-1}\Big|\frac{d}{ds}\mathfrak{u}^*(\epsilon s)\Big|ds,\\&=\sqrt{2}\int_0^{s_1}\sqrt{V(\bar{u}(s))}ds=\mathcal{J}_V(\bar{u},(0,s_1))\\&\lim_{\epsilon\to0^+}\epsilon^{-2}\mathcal{J}_U(v(\epsilon\cdot),(1+\sigma,s_1+\sigma))=\lim_{\epsilon\to0^+}\sqrt{2}\int_{1+\sigma}^{s_1+\sigma}\sqrt{\epsilon^{-2}U(v(\epsilon s))}\epsilon^{-1}\Big|\frac{d}{ds}v(\epsilon s)\Big|ds,\\&=\sqrt{2}\int_1^{s_1}\sqrt{V(\bar{v}(s))}\sqrt{1+g'^2(s)}ds=\mathcal{J}_V(\bar{v},(1,s_1)).\end{aligned}$$

This and (iii) above imply that, indeed, the inequality (2.36) holds for small $\epsilon>0$. The proof is complete. □

We can now complete the proof of Theorem 1.1. We show that the map $u^* : (T_-, T_+) \to \mathbb{R}^n$ possesses all the required properties. The fact that u^* satisfies (1.2) and (1.4) follows from Lemma 2.3. Lemma 2.2 implies (1.5) and, if $T_- > -\infty$, also (1.6). The fact that $x_- \in \Gamma_- \setminus P$ is a consequence of Lemma 2.4 and implies that Γ_- has positive diameter. Viceversa, if Γ_- has positive diameter, Lemmas 2.5 and 2.6 imply that $T_- > -\infty$ and that (1.6) holds for some $x_- \in \Gamma_- \setminus P$. The proof of Theorem 1.1 is complete.

Remark. From Theorem 1.1 it follows that if N is even then there are at least $N/2$ distinct orbits connecting different elements of $\{\Gamma_1, \dots, \Gamma_N\}$. If N is odd there are at least $(N+1)/2$. Simple examples show that, given distinct $\Gamma_i, \Gamma_j \in \{\Gamma_1, \dots, \Gamma_N\}$, an orbit connecting them does not always exist. Let

$$\mathcal{U}_{ij} = \{u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : u((T_-^u, T_+^u)) \subset \Omega, u(T_-^u) \in \Gamma_i, u(T_+^u) \in \Gamma_j\}$$

with $i \neq j$ and

$$d_{ij} = \inf_{u \in \mathcal{U}_{ij}} \mathcal{A}(u, (T_-^u, T_+^u)).$$

An orbit connecting Γ_i and Γ_j exists if

$$d_{ij} < d_{ik} + d_{kj}, \quad \forall k \neq i, j.$$

The proof of Theorem 1.2 uses, with obvious modifications, the same arguments as in the proof of Theorem 1.1 to characterize u^* as the limit of a minimizing sequence $\{u_j\}$ of the action functional

$$\mathcal{A}(u, (0, T^u)) = \int_0^{T^u} \Big(\frac{1}{2}|\dot{u}(t)|^2 + U(u(t))\Big) dt. \tag{2.39}$$

in the set

$$\mathcal{U} = \{u \in W^{1,2}((0, T^u); \mathbb{R}^n) : 0 < T_+^u < +\infty,\ u(0) = 0,\ u([0, T_+^u)) \subset \Omega,\ u(T_+^u) \in \partial\Omega\}. \tag{2.40}$$

Remark. In the symmetric case of Theorem 1.2 it is easy to construct an example with $T_+ < T_+^\infty$. For $U(x) = 1 - |x|^2$, $x \in \mathbb{R}^2$, the solution $u : [0, \pi/2] \to \mathbb{R}^2$ of (1.2) determined by (1.4) and $u([0, \pi/2]) = \{(s, 0) : s \in [0, 1]\}$ is a minimizer of $\mathcal{A}$ in $\mathcal{U}$. For ϵ small, let $t_\epsilon = \arcsin(1-\epsilon)$ and define $u_\epsilon : [0, T^{u_\epsilon}] \to \mathbb{R}^2$ as the map determined by (1.4), $u_\epsilon([0, t_\epsilon]) = \{(s, 0) : s \in [0, 1-\epsilon)\}$ and $u_\epsilon((t_\epsilon, T^{u_\epsilon}]) = \{(1-\epsilon, s) : s \in (0, \sqrt{2\epsilon - \epsilon^2}]\}$. In this case $T_+ = \pi/2$ and $T_+^\infty = 3\pi/4$.

2.1 On the existence of heteroclinic connections

Corollary 1.3 states the existence of heteroclinic connections under the assumptions of Theorem 1.1 and, in particular, that $U \in C^2$. Actually, by examining the proof of Theorem 1.1 we can establish an existence result under weaker hypotheses. In the special case $\partial\Omega = P$, $\#P \geq 2$, given $p_- \in P$, the set $\mathcal{U}$ defined in (2.1) takes the form

$$\begin{aligned}\mathcal{U} = \big\{&u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty,\\ &u((T_-^u, T_+^u)) \subset \Omega,\ U(u(0)) = U_0,\ u(T_-^u) = p_-,\ u(T_+^u) \in P \setminus \{p_-\}\big\}.\end{aligned}$$

In this section we slightly enlarge the set $\mathcal{U}$ by allowing $T_\pm^u = \pm\infty$ and consider the admissible set

$$\begin{aligned}\widetilde{\mathcal{U}} = \big\{&u \in W_{loc}^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty \leq T_-^u < T_+^u \leq +\infty,\\ &u((T_-^u, T_+^u)) \subset \Omega,\ U(u(0)) = U_0,\ \lim_{t \to T_-^u} u(t) = p_-,\ \lim_{t \to T_+^u} u(t) \in P \setminus \{p_-\}\big\}.\end{aligned}$$

Proposition 2.7. *Assume that U is a non-negative continuous function, which vanishes in a finite set P, $\#P \geq 2$, and satisfies*

$$\sqrt{U(x)} \geq \sigma(|x|), \;\; x \in \Omega, \;\; |x| \geq r_0$$

for some $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{+\infty} \sigma(r)dr = +\infty$.

Given $p_- \in P$ there is $p_+ \in P \setminus \{p_-\}$ and a Lipschitz-continuous map $u^* : (T_-, T_+) \to \Omega$ that satisfies (1.4) almost everywhere on (T_-, T_+),

$$\lim_{t \to T_\pm} u^*(t) = p_\pm,$$

and minimizes the action functional $\mathcal{A}$ on $\tilde{\mathcal{U}}$.

Proof. We begin by showing that

$$a_0 = \inf_{u \in \mathcal{U}} \mathcal{A} = \inf_{u \in \tilde{\mathcal{U}}} \mathcal{A} = \tilde{a}_0. \tag{2.41}$$

Since $\mathcal{U} \subset \tilde{\mathcal{U}}$ we have $a_0 \geq \tilde{a}_0$. On the other hand arguing as in the proof of Lemma 2.2, if $T_+ - T_- = +\infty$, given a small number $\epsilon > 0$, we can construct a map $u_\epsilon \in \mathcal{U}$ that satisfies

$$a_0 \leq \mathcal{A}(u_\epsilon, (T_-^{u_\epsilon}, T_+^{u_\epsilon})) \leq \mathcal{A}(u, (T_-^u, T_+^u)) + \eta_\epsilon$$

where $\eta_\epsilon \to 0$ as $\epsilon \to 0$. This implies $a_0 \leq \tilde{a}_0$ and establishes (2.41). It follows that we can proceed as in the proof of Theorem 1.1 and define $u^* \in \tilde{\mathcal{U}}$ as the limit of a minimizing sequence $\{u_j\} \subset \mathcal{U}$. The arguments in the proof of Lemma 2.2 show that (2.8) holds. It remain to show that u^* is Lipschitz-continuous. Looking at the proof of Lemma 2.3 we see that the continuity of U is sufficient for establishing that (1.4) holds almost everywhere on (T_-, T_+), and the Lipschitz character of u^* follows. The proof is complete. □

Remark. Without further information on the behavior of U in a neighborhood of $p_\pm$ nothing can be said on $T_\pm$ being finite or infinite and it is easy to construct examples to show that all possible combinations are possible. As shown in Lemma 2.4 a sufficient condition for $T_\pm = \pm\infty$ is that, in a neighborhood of $p = p_\pm$, $U(x)$ is bounded by a function of the form $c|x - p|^2$, $c > 0$. U of class C^1 is a sufficient condition in order that u^* is of class C^2 and satisfies (1.2).

3 Examples

In this section we show a few simple applications of Theorems 1.1 and 1.2.
Our first application describes a class of potentials with the property that, in spite of the existence of possibly infinitely many critical values, (1.2) has a nontrivial periodic orbit on any energy level.

Proposition 3.1. *Assume that $U : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\begin{aligned} &U(-x) = U(x), \;\; x \in \mathbb{R}^n, \\ &U(0) = 0, \; U(x) < 0 \text{ for } x \neq 0, \\ &\lim_{|x| \to \infty} U(x) = -\infty \end{aligned}$$

Assume moreover that each non zero critical point of U is hyperbolic with Morse index $i_m \geq 1$. Then there is a nontrivial periodic orbit of (1.2) on the energy level $\frac{1}{2}|\dot{u}|^2 - U(u) = \alpha$ for each $\alpha > 0$.

Proof. For each $\alpha > 0$ we set $\tilde{U} = U(x) + \alpha$ and let $\Omega \subset \{\tilde{U} > 0\}$ be the connected component that contains the origin. Ω is open, nonempty and bounded and, from the assumptions on the properties of the critical points of U, it follows that $\partial\Omega$ is connected and contains at most a finite number of critical points. Therefore we are under the assumptions of Corollary 1.6 for the case $N = 1$ and the existence of the periodic orbit follows. □

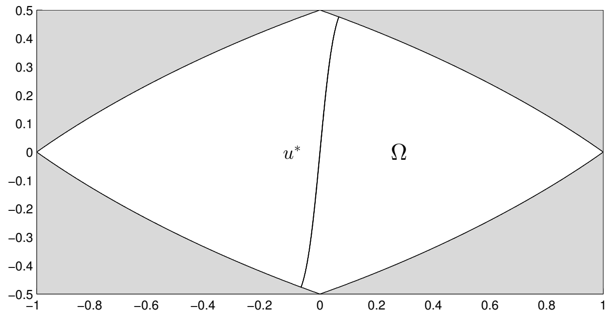

Figure 3: Symmetric periodic orbit for the example with potential (3.1).

An example of potential $U : \mathbb{R}^2 \to \mathbb{R}$ that satisfies the assumptions in Proposition 3.1 is, in polar coordinates r, θ,

$$U(r,\theta) = -r^2 + \frac{1}{2}\tanh^4(r)\cos^2(r^{-1})\cos^{2k}(2\theta),$$

where $k > 0$ is a sufficiently large number.

Next we give another application of Corollary 1.6. For the potential $U : \mathbb{R}^2 \to \mathbb{R}$, with

$$U(x) = \frac{1}{2}(1 - x_1^2)^2 + \frac{1}{2}(1 - 4x_2^2)^2, \tag{3.1}$$

the energy level $\alpha = -\frac{1}{2}$ is critical and corresponds to four hyperbolic critical points $p_1 = (1, 0)$, $-p_1$, $p_2 = (0, \frac{1}{2})$ and $-p_2$. The connected component $\Omega \subset \{\tilde{U} > 0\}$, ($\tilde{U} = U(x) - \frac{1}{2}$) that contains the origin is bounded by a simple curve Γ that contains $\pm p_1$ and $\pm p_2$. In spite of the presence of these critical points, from Theorem 1.2 it follows that there is a minimizer $u \in \mathcal{U}$, with $\mathcal{U}$ as in (2.40) and $u(T^u) \in \Gamma \setminus \{\pm p_1, \pm p_2\}$, and Corollary 1.6 implies the existence of a periodic solution v^*. Note that there are also two heteroclinic orbits, solutions of (1.2) and (1.4):

$$u_1(t) = (\tanh(t), 0), \qquad u_2(t) = (0, \frac{1}{2}\tanh(2t)).$$

These orbits connect p_j to $-p_j$, for $j = 1, 2$. By Theorem 1.2 both u_1 and u_2 have action greater than $v^*|_{(-T_+, T_+)}$.

Our last example shows that Theorems 1.1 and 1.2 can be used to derive information on the rich dynamics that (1.2) can exhibit when U undergoes a small perturbation. We consider a family of potentials $U : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}$. We assume that $U(x, 0) = x_1^6 + x_2^2$ which from various points of view is a structurally unstable potential and, for $\lambda > 0$ small, we consider the perturbed potential

$$U(x, \lambda) = 2\lambda^4 x_1^2 + x_2^2 - 2\lambda^2 x_1 x_2 - 3\lambda^2 x_1^4 + x_1^6. \tag{3.2}$$

This potential satisfies $U(-x, \lambda) = U(x, \lambda)$ and, for $\lambda > 0$, has the five critical points $p_0, \pm p_1$ and $\pm p_2$ defined by

$$\begin{aligned} p_0 &= (0, 0), \\ p_1 &= (\lambda(1 - (\tfrac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 - (\tfrac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}), \\ p_2 &= (\lambda(1 + (\tfrac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 + (\tfrac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}), \end{aligned}$$

which are all hyperbolic.

We have $U(p_2, \lambda) < 0 = U(p_0, \lambda) < U(p_1, \lambda)$ and p_0 is a local minimum, p_1 a saddle and p_2 a global minimum. Let α be the energy level. For $-\alpha < U(p_2, \lambda)$ or $-\alpha \geq U(p_1, \lambda)$ no information can be

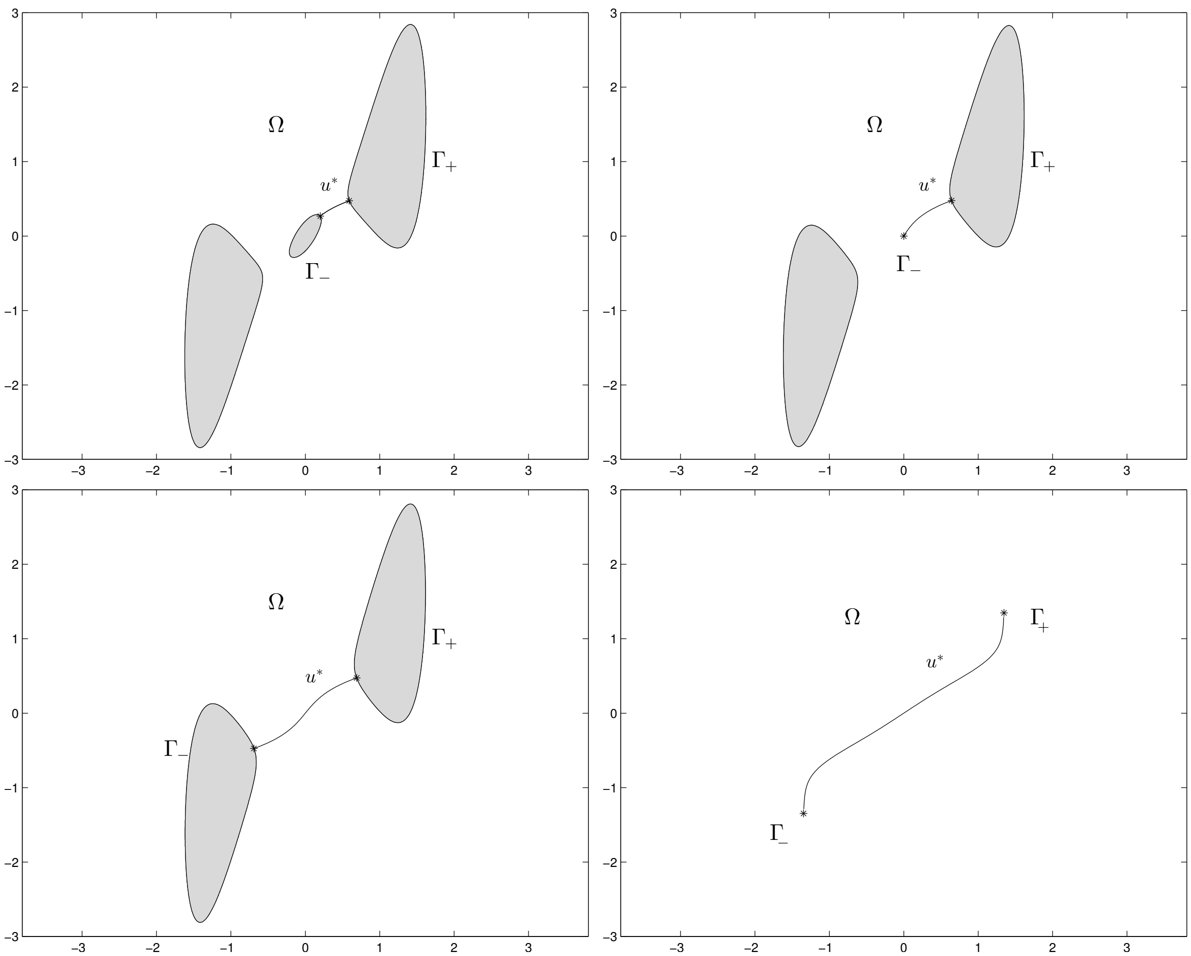

Figure 4: Bifurcations of dynamics of (1.2) with the $\alpha = 0$, bottom left: $\alpha = 0.05$, bottom right: $\alpha = -U(p_2, 1)$. The shaded regions are not accessible.

derived from Theorems 1.1 and 1.2 therefore we assume $-\alpha \in [U(p_2, \lambda), U(p_1, \lambda))$. For $-\alpha = U(p_2, \lambda)$ Corollary 1.3 or Corollary 1.6 yields the existence of a heteroclinic connection u_2 between $-p_2$ and p_2. For $-\alpha \in (U(p_2, \lambda), 0)$ Corollary 1.6 implies the existence of a periodic orbit u_α. This periodic orbit converges uniformly in compact intervals to u_2 and the period $T_\alpha \to +\infty$ as $-\alpha \to U(p_2, \lambda)^+$. For $\alpha = 0$ Corollary 1.4 implies the existence of two orbits u_0 and $-u_0$ homoclinic to $p_0 = 0$. We can assume that u_0 satisfies the condition $u_0(-t) = u_0(t)$ and that $u_\alpha(0) = 0$. Then we have that $u_\alpha(\cdot \pm \frac{T_\alpha}{4})$ converges uniformly in compact intervals to $\mp u_0$ and $T_\alpha \to +\infty$ as $-\alpha \to 0^-$. For $-\alpha \in (0, U(p_1, \lambda))$, $\partial\Omega$ is the union of three simple curves all of positive diameter: Γ_0 that includes the origin and $\pm\Gamma_2$ which includes $\pm p_2$ and Corollary 1.5 together with the fact that $U(\cdot, \lambda)$ is symmetric imply the existence of two periodic solutions $\tilde{u}_\alpha$ and $-\tilde{u}_\alpha$ with $\tilde{u}_\alpha$ that oscillates between Γ_0 and Γ_2 in each time interval equal to $\frac{T_\alpha}{2}$. Assuming that $\tilde{u}_\alpha(0) \in \Gamma_2$ we have that, as $-\alpha \to 0^+$, $\tilde{u}_\alpha \to u_0$ uniformly in compacts and $T_\alpha \to +\infty$. Finally we observe that, in the limit $-\alpha \to U(p_1, \lambda)^-$, $\tilde{u}_\alpha$ converges uniformly in $\mathbb{R}$ to the constant solution $u \equiv p_1$.

Acknowledgements

The first author is indebted with Peter Bates for fruitful discussions on the subject of this paper.

References

[1] N. Alikakos and G. Fusco. On the connection problem for potentials with several global minima. *Indiana Univ. Math. Journ.* **57** No. 4, 1871-1906 (2008)

[2] P. Antonopoulos and P. Smyrnelis. On minimizers of the Hamiltonian system $u'' = \nabla W(u)$, and on the existence of heteroclinic, homoclinic and periodic connections. Preprint (2016)

[3] A. Braides. Approximation of Free-Discontinuity Problems. Lectures Notes in Mathematics 1694, Springer-Verlag, Heidelberg (1998)

[4] A. Bressan. Tutorial on the Center Manifold Theorem. Hyperbolic systems of balance laws. CIME course (Cetraro 2003). Springer Lecture Notes in Mathematics 1911, 327-344. Springer-Verlag, Heidelberg (2007)

[5] G. Buttazzo, M. Giaquinta, S. Hildebrandt. One-dimensional Calculus of Variations: an Introduction Oxford University Press, Oxford (1998)

[6] N. V. De Paulo and P. A. S. Salomão. Systems of transversal sections near critical energy levels of Hamiltonian systems in $\mathbb{R}^4$, arXiv:1310.8464v2 (2016)

[7] A. Monteil and F. Santambrogio. Metric methods for heteroclinic connections. *Mathematical Methods in the Applied Sciences*, DOI: 10.1002/mma.4072 (2016)

[8] C. Sourdis. The heteroclinic connection problem for general double-well potentials. *Mediterranean Journal of Mathematics* **13** No. 6, 4693-4710 (2016)

[9] P. Sternberg. Vector-Valued Local Minimizers of Nonconvex Variational Problems, *Rocky Mountain J. Math.* **21** No. 2, 799-807 (1991)

[10] A. Vanderbauwhede. Centre manifolds, normal forms and elementary bifurcations. *Dynamics Reported* **2** No. 4, 89-169 (1989)

[11] A. Zuniga and P. Sternberg. On the heteroclinic connection problem for multi-well gradient systems. *Journal of Differential Equations* **261** No. 7, 3987-4007 (2016)