TV denoising of two balls in the plane

Vicent Caselles[*], Matteo Novaga[†], Christiane Pöschl [‡]

Abstract

The aim of this paper is to compute the explicit solution of the total variation denoising problem corresponding to the characteristic function of a set which is the union of two planar disjoint balls with different radii.

1 Introduction

The purpose of this paper is to compute explicit solutions of the total variation denoising problem

$$\min_{u\in BV(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |Du| + \frac{1}{2\lambda}\int_{\mathbb{R}^2} |u-f|^2 \, dx \, , \tag{1}$$

where $f = \chi_S$ and $S \in \mathbb{R}^2$ is the union of two balls whose interiors are disjoint sets and $\lambda > 0$.

The study of explicit solutions of (1) was initiated in [8, 9], where the authors studied the bounded sets of finite perimeter S in $\mathbb{R}^2$ for which the solution of (1) is a multiple of χ_S. Such sets, which were called calibrable, produce solutions of the total variation flow which evolve at constant speed without distortion of the boundary. They where characterized in [8] by the existence of a vector field $\xi \in L^\infty(\mathbb{R}^2,\mathbb{R}^2)$ such that $|\xi| \leq 1$, $\xi \cdot D\chi_S = |D\chi_S|$, and $-\mathrm{div}\,\xi = \frac{P(S)}{|S|}\chi_S$, where $P(S)$ denotes the perimeter of S and $|S|$ denotes the area of S. For bounded connected sets $S \subset \mathbb{R}^2$, calibrable sets are characterized as the convex, $C^{1,1}$ sets satisfying the bound $\mathrm{ess\,sup}_{x\in\partial S}\kappa(x) \leq \frac{P(S)}{|S|}$, where $\kappa(x)$ denotes the curvature of ∂S at the point x. The paper [8] gives also a characterization of non connected calibrable sets in $\mathbb{R}^2$. The paper [9] describes the explicit solution of (1) for sets $S \subset \mathbb{R}^2$ of the form $S = C_0 \setminus \cup_{i=1}^k C_i$ where the sets C_i, $i = 0, 1, \ldots, k$, are convex and satisfy some bounds on the curvature of its boundary. The explicit solution when the set S is a convex subset of $\mathbb{R}^2$ was described in [4] (also the case of a set S which is a union of convex sets which are sufficiently far from each other). The explicit solution corresponding to a general convex set in $\mathbb{R}^N$ was described in the

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[*] DTIC, Universitat Pompeu–Fabra, Roc Boronat 138, 08018 Barcelona, Spain

[†] Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5 56127 Pisa, Italy, e–mail: novaga@dm.unipi.it

[‡] Institute of Mathematics, Alpen Adria University Klagenfurt, Universitätsstraße 65, 9020 Klagenfurt e–mail: Christiane.Poeschl@aau.at

papers [3, 11, 2] (covering also the case of the union of convex sets which are sufficiently far in a precise sense).

When S is the union of two convex sets S_1, S_2, the situation may become more complicated, even when both sets are calibrable. In particular, if the sets are near each other, more precisely if the perimeter of its convex hull is smaller than the sum of the perimeters of the two sets, then the sets interact, and the solution outside S is not null (for small values of λ). This is the case, for instance, when S is the union of two balls in $\mathbb{R}^2$, which is the object of this paper.

Let us mention in this context the work of Allard [1] who calculated the solution of (1) when S is the union of two balls with the same radius. He also computed the solution of (1) when S is the union of two squares. In this paper we extend the result of Allard [1] to the case where S is the union of any two balls in $\mathbb{R}^2$. The interesting case is when the perimeter of the convex hull of S is less than the perimeter of S, since otherwise the solution can be described as the sum of the solutions corresponding to each ball [8]. Our approach differs from the one in [1] even for the case of two balls of the same radius. While the solution in [1] is obtained by a explicit computation, we describe it in a shorter way by means of more general geometric arguments. Our starting point is the observation that u_λ is a solution of (1) if and only if the sets $[u_\lambda \geq s]$ minimize the variational problem [12, 1]

$$\mathcal{F}_{s,\lambda}(X) := P(X) + \frac{s}{\lambda}\,|X \setminus S| - \frac{(1-s)}{\lambda}\,|X \cap S| \qquad s \in [0,1], \lambda > 0\ , \tag{2}$$

where $P(X)$ is the perimeter of X (and we understand that $P(X) = +\infty$ if $\chi_X \notin BV(\mathbb{R}^2)$). Let us point out that, for $\lambda > 0$ fixed, the solutions of (2) are monotonically decreasing as s increases, and can be then packed together to build up a function which solves (1) [3, 12]. Thus, to compute the solution of (1) we study the solution of (2) and those solutions can be constructed by means of geometric arguments.

On one hand, the Euler-Lagrange equation of (2) tells us that, if $C_{s,\lambda}$ is a minimizer of (2), then $\partial C_{s,\lambda}$ is $\mathcal{C}^{1,1}$, $\partial C_{s,\lambda}$ has curvature $\frac{1-s}{\lambda}$ inside S and $-\frac{s}{\lambda}$ outside S. When S is the union of two disjoint open balls S_1, S_2 and $\lambda \leq r_c$, for some value of $r_c > 0$ that depends on the geometry of S, we prove that the intersection of $S_i, i = 1, 2$, with the minimizing sets is either S_i or $\emptyset$. We also give a counterexample showing that this result is not true for any value of λ. Thus, for $\lambda \leq r_c$, the possible minimizers of (2) are: $\emptyset, S_1, S_2, S, \mathit{Close}_{\frac{\lambda}{s}}(S)$, where $\mathit{Close}_r(S)$ denotes the r-closing of the set S, that is, the complement of the union of the balls of radius r contained in $\mathbb{R}^2 \setminus S$.

The computation of explicit examples of TV denoising permits to exhibit qualitative features of the solution. In particular, the appearance of new level lines is a undesirable feature for denoising. Better denoising algorithms have been developed in the last few years [10].

Let us describe the plan of the paper. In Section 2 we review some known results that permit to set the context of our analysis.

In Section 3 we describe the generic properties of the minimizers $C_{s,\lambda}$ of (2) and we prove that, if S is the union of two balls and $\lambda \leq r_c$, then the intersection of $C_{s,\lambda}$ with S_i, $i = 1, 2$, is either S_i or the empty set.

This permits to reduce the set of possible minimizers of (4) to the following six ones: $\emptyset, S_1, S_2, S, Close_{\frac{\lambda}{s}}(S), \Gamma_{s,\lambda}(S)$. In Section 5 we explain how to construct the sets $\Gamma_{s,\lambda}(S)$ as well as how to construct the explicit solutions for (1). The proof of the main theorem can be found in Section 6. In Section 4 we describe how to calculate the dual norm of the function χ_S.

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This paper was inspired by our coauthor and friend Vicent Caselles. His passion and his strong motivation were a continuous stimulus in our research, and we dedicate this work to his memory.

2 Preliminaries

2.1 Total variation and perimeter

Let Ω be an open subset of $\mathbb{R}^2$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup \left\{ \int_\Omega u \, \mathrm{div}\, z \; dx : z \in C_0^\infty(\Omega; \mathbb{R}^2), \|z\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |z(x)| \leq 1 \right\},$$

and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. $BV(\Omega)$ is a Banach space when endowed with the norm $\int_\Omega |u| \; dx + |Du|(\Omega)$.

Let us denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

A measurable set $E \subseteq \mathbb{R}^2$ is said to have finite perimeter if $\chi_E \in BV(\mathbb{R}^2)$. The perimeter of E is defined as $P(E) := |D\chi_E|(\mathbb{R}^2)$. We recall that when E is a finite-perimeter set with regular boundary (for instance, Lipschitz), its perimeter $P(E)$ also coincides with the more standard definition $\mathcal{H}^1(\partial E)$. For more properties and references on functions of bounded variation we refer to [5]. We also mention the following review papers on applications to image analysis and denoising. [10, 14, 13].

2.2 Morphological operators

Definition 2.1 (Opening and Closing operators). *For any set X and $r > 0$, let $B_r(x)$ be a ball with radius r and center x. We define the opening and the closing of X, with radius r,* *respectively by*

$$\begin{aligned} Open_r(X) &:= \bigcup_{x : B_r(x) \subset X} B_r(x) \ , \\ Close_r(X) &:= \left(Open_r\left(X^C\right)\right)^C \ , \end{aligned}$$

where X^C denotes the complement of the set X.

The opening operator is anti-extensive ($Open_r(X) \subset X$), conserves the subset property ($X \subset Y$ then $Open_r(X) \subset Open_r(Y)$) and is idempotent ($Open_r(X) = Open_r(Open_r(X))$). For more on application of morphological operators we refer to [18]. Later we need the following properties of the opening and closing operator.

Lemma 2.2 (Properties of the Opening and Closing operator)**.** *Let S be an arbitrary set. The curvature of $\partial Close_r(X)$ is larger or equal to $-\frac{1}{r}$, the curvature of $\partial Open_r(X)$ is less or equal to $\frac{1}{r}$. Consequently the curvatures of $\partial Close_r(X) \setminus X, \partial Open_r(S) \cap S$ are $-\frac{1}{r}, \frac{1}{r}$ respectively. Moreover, if $Close_r(X) \neq X$, then $\min\{\kappa(\partial X)\} < \frac{1}{r}$. If $Open_r(X) \neq X$, then $\max\{\kappa(\partial X)\} > \frac{1}{r}$.*

Proof. Assume that there exists a point $A \in Open_r(X)$, with curvature $> \frac{1}{r}$, then there exists no circle touching $Open_r(X)$ at A that lies inside $Open_r(X)$, this contradicts the definition of the opening operator, hence we can conclude that the curvature of $\partial Open_r(X)$ is smaller or equal to $\frac{1}{r}$. The proof of the second estimate is analog. ☐

2.3 Review of some basic results

The following result was proved in [1, 12].

Proposition 2.3. *Let $S \subset \mathbb{R}^2$ be a bounded measurable set. Then there is a unique solution u_λ of (1), which satisfies the Euler-Lagrange equation*

$$u_\lambda - \lambda \operatorname{div} z = \chi_S \qquad \text{in } \mathbb{R}^2, \tag{3}$$

where $z : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $\|z\|_\infty \leq 1$ and $z \cdot Du_\lambda = |Du_\lambda|$.

Moreover, for any $s \in \mathbb{R}$, $\{u_\lambda \geq s\}$ (resp. $\{u_\lambda > s\}$) is the maximal (resp. the minimal) solution of

$$\min_{X \subset \mathbb{R}^2} \mathcal{F}_{s,\lambda}(X) := P(X) + \frac{s}{\lambda}|X \setminus S| - \frac{(1-s)}{\lambda}|X \cap S|. \tag{4}$$

In particular, for all t but a countable set the solution of (4) is unique.

Conversely, for any $s \in \mathbb{R}$, let Q_s be a solution of (4). If $s > s'$, then $Q_s \subseteq Q_{s'}$. The function

$$u(x) = \sup\{s : x \in Q_s\}$$

is the solution of (1).

Thus, in order to build up the solution of (1) it suffices to compute the solutions of the family of problems (4). This will be the strategy we follow to compute the explicit solution when S is the union of two balls.

Recall that if $g \in L^2(\mathbb{R}^2)$ the dual BV-norm of g is given by

$$\|g\|_* = \sup_{u \in BV(\mathbb{R}^2), |Du|(\mathbb{R}^2) \leq 1} \int_{\mathbb{R}^2} gu \, dx.$$

Then $\|g\|_* \leq 1$ if and only if

$$\int_{\mathbb{R}^2} gu \, dx \leq \int_{\mathbb{R}^2} |Du|$$

for any $u \in BV(\mathbb{R}^2)$. This is equivalent to say that

$$\int_F g \leq P(F), \qquad \text{for any set } F \text{ of finite perimeter.}$$

Let us first recall a result that permits to compute the value of λ for which the solution $u_\lambda = 0$. The result was proved in [17, 8].

Proposition 2.4. *([8]) Let $g \in L^2(\mathbb{R}^2)$. Let us consider the problem:*

$$\min_{u \in BV(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)} \mathcal{F}_\lambda(g) \tag{5}$$

where

$$\mathcal{F}_\lambda(g) := \int_{\mathbb{R}^2} |Du| + \frac{1}{2\lambda} \int_{\mathbb{R}^2} |u - g|^2 \, dx. \tag{6}$$

The following conditions are equivalent

- (i) $u = 0$ *is a solution of* (5).
- (ii) $\|g\|_* \leq \lambda$
- (iii) *There is a vector field* $\xi \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $\|\xi\|_\infty \leq 1$ *such that* $-\mathrm{div}\, \xi = g$.

The following result has been proved in [16, 8, 4, 15].

Theorem 2.5. *Let $C \subset \mathbb{R}^2$ be a bounded set of finite perimeter, and assume that C is connected. Let $\gamma > 0$. The following conditions are equivalent:*

- *(i) C decreases at speed γ, i.e, for any $\lambda > 0$ $u_\lambda := (1 - \lambda\gamma)^+ \chi_C(x)$ is the solution of (1) corresponding to $\chi_C(x)$.*
- *(ii) C is convex, $\gamma = \gamma_C := \frac{P(C)}{|C|}$ and minimizes the functional*

$$\mathcal{G}_{\gamma_C}(X) := P(X) - \gamma_C |X|, \qquad X \subseteq C, \; X \; of \; finite \; perimeter.$$

(iii) C is convex, ∂C is of class $C^{1,1}$, $\gamma = \gamma_C := \frac{P(C)}{|C|}$, and the following inequality holds:

$$\operatorname*{ess\,sup}_{p\in\partial C} \kappa_{\partial C}(p) \leq \gamma_C,$$

where $\kappa_{\partial C}(p)$ denotes the curvature of ∂C at the point p.

For all $r \in \mathbb{R}$, we set $r^+ := \max\{0, r\}$. The following result has been proved in [8, Theorem 7 and Proposition 8].

Lemma 2.6. *Let $S_1, S_2 \subset \mathbb{R}^2$ be two disjoint balls, let $S = S_1 \cup S_2$ and $f = \chi_S$. Then*

$$u_\lambda = \left(1 - \frac{P(S_1)}{|S_1|}\lambda\right)^+ \chi_{S_1} + \left(1 - \frac{P(S_2)}{|S_2|}\lambda\right)^+ \chi_{S_2}$$

is a solution of (1) for any $\lambda > 0$ if and only if

$$P(S) \leq P(\mathrm{co}(S)) \,, \tag{7}$$

where $\mathrm{co}(S)$ denotes the convex envelope of S. In other words, the solution of (1) is the sum of the two solutions corresponding to χ_{S_1} and χ_{S_2} if and only if (7) holds.

In the general case the minimizers of $\mathcal{F}_{s,\lambda}$ can be subsets of S or contain parts outside S, as we shall see in the following section.

3 Properties of minimizers

As explained in Section 2.3 our purpose is to characterize the minimizers of $\mathcal{F}_{s,\lambda}$ when S is the union of two balls S_1, S_2 with disjoint interiors and distance d. In order to fix the notation we assume S_1, S_2 are open balls of radii $r_1 \geq r_2$.

Let us first state a simple geometric result which will be useful in the proof of Proposition 3.2 below.

Lemma 3.1. *Let B_1, B_2, B_3 be three open balls of equal radius, intersecting ∂S_2 at equal angles. Let Γ_{S_2} be the arc of ∂S_2 contained in $\mathrm{co}(S)$. Assume the three balls intersect Γ_{S_2} and B_3 is between B_1 and B_2 when we go along Γ_{S_2} (see Figure 1). If S_1 intersects B_1 and B_2, then it intersects also B_3. The same statement holds interchanging S_1 and S_2.*

Proof. Observe that the centers of B_1, B_2, and B_3, denoted respectively by q_1, q_2, q_3, are contained in a circle concentric with S_2. Let p be the center of S_1 and r be the common radius of B_i, $i = 1, 2, 3$. Consider the triangle formed by the segments $[p, q_1]$, $[p, q_2]$ and $[q_1, q_2]$. Notice that since S_1 intersects B_1 and B_2, $|p - q_1| \leq r_1 + r$ and $|p - q_2| \leq r_1 + r$. Since q_3 is contained in the interior of such triangle then $|p - q_3| < r_1 + r$, and therefore S_1 intersects B_3. $\square$

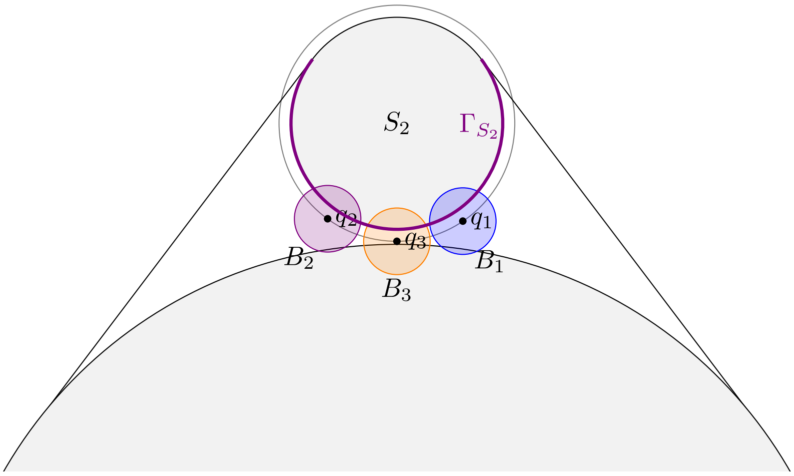

Figure 1: The construction in the proof of Lemma 3.1.

Proposition 3.2. *Let $C_{s,\lambda}$ be a minimizer of $\mathcal{F}_{s,\lambda}$. Then the boundary $\partial C_{s,\lambda}$ is of class $C^{1,1}$, $C_{s,\lambda} \subset \overline{\mathrm{co}(S)}$, and one of the following possibilities holds:*

a) $C_{s,\lambda} \in \{\emptyset, S_1, S, Close_{\frac{\lambda}{s}}(S)\}$, and $C_{s,\lambda} \neq S_2$ if $r_1 > r_2$;

b) $S_1 \subset C_{s,\lambda}$, $\partial C_{s,\lambda} \cap S_2$ is a circular arc with curvature $\frac{1-s}{\lambda}$, and $\partial C_{s,\lambda} \setminus \overline{S}$ is composed by two circular arcs with curvature $-\frac{s}{\lambda}$.

Proof. The regularity of $\partial C_{s,\lambda}$ is a classical result [6]. The Euler-Lagrange equations say that, if non-empty, $\partial C_{s,\lambda} \setminus \overline{S}$ are arcs of circle of curvature $-\frac{s}{\lambda}$, and $\partial C_{s,\lambda} \cap S$ are arcs of circle of curvature $\frac{1-s}{\lambda}$. In particular, $\partial C_{s,\lambda}$ has finitely many connected components which are $C^{1,1}$ Jordan curves, and any two of them have positive distance.

Notice that the energy is additive on the connected components, that is, if $\mathcal{CC}_{s,\lambda}$ denotes the set of connected components of $C_{s,\lambda}$, then $\mathcal{F}_{s,\lambda}(C_{s,\lambda}) = \sum_{C \in \mathcal{CC}_{s,\lambda}} \mathcal{F}_{s,\lambda}(C)$. Moreover $\mathcal{F}_{s,\lambda}(C) \leq 0$ for any $C \in \mathcal{CC}_{s,\lambda}$, otherwise we can eliminate this component decreasing the energy. Let C be a connected component of $C_{s,\lambda}$. Modulo null sets, if $C \cap S_i = \emptyset$, $i \in \{1,2\}$, then $C \subseteq S_j$, $j \in \{1,2\}$, $j \neq i$. Otherwise, by replacing C by $C \cap S_j$ we decrease the energy of $C_{s,\lambda}$. Thus there are only three possibilities: $C \subseteq S_1$, $C \subseteq S_2$, or $C \cap S_1 \neq \emptyset$ and $C \cap S_2 \neq \emptyset$.

Without loss of generality, we can assume that $\mathrm{dist}(S_1, S_2) > 0$. Having proved the result in this case, by passing to the limit we get it also when $\mathrm{dist}(S_1, S_2) = 0$. We divide the rest of the proof in several steps. Without loss of generality we may assume that $C_{s,\lambda}$ is an open set.

Step 1. If $r_1 > r_2$ then $C_{s,\lambda} \neq S_2$.

Assume by contradiction that $C_{s,\lambda} = S_2$, then $\mathcal{F}_{s,\lambda}(C_{s,\lambda}) = \mathcal{F}_{s,\lambda}(S_2) = 2\pi r_2 - \frac{(1-s)\pi}{\lambda} r_2^2 \leq 0$, which implies $s < 1$ and $r_2 \geq \frac{2\lambda}{1-s}$. This in turn implies that $\mathcal{F}_{s,\lambda}(S_1) < \mathcal{F}_{s,\lambda}(S_2)$, contradicting the minimality of $C_{s,\lambda}$.

Step 2. We have $C_{s,\lambda} \subset \mathrm{co}(S)$.

Being $\mathrm{co}(S)$ convex, this follows from the fact that $C_{s,\lambda} \cap \mathrm{co}(S)$ has lower energy than $C_{s,\lambda}$, with equality if and only if $C_{s,\lambda} \subset \mathrm{co}(S)$.

Step 3. Let C be a connected component of $C_{s,\lambda}$ intersecting only one of the two circles, say S_i, then $C = S_i$.

Replacing C with $C \cap S_i$ decreases the energy, hence we may assume $C \subset S_i$. On the other hand, C does not have holes since by filling them we would also decrease the energy. Since ∂C is $C^{1,1}$, then C is a ball of radius $r(s) = \frac{\lambda}{1-s}$. As we observed at the beginning of the proof, it is at positive distance from the other connected components. Thus we may dilate it to a ball B_r of radius r contained in S_i. Since $\mathcal{F}_{s,\lambda}(B_r) = 2\pi r - \frac{1-s}{\lambda}\pi r^2$, for $r > r(s)$ near $r(s)$ we have $\mathcal{F}_{s,\lambda}(B_r) < \mathcal{F}_{s,\lambda}(C)$ and this permits to decreases the energy of $C_{s,\lambda}$. Thus $C = S_i$.

Step 4. Let Γ be a connected component of $\partial C_{s,\lambda}$, and assume that $\Gamma \setminus \overline{S}$ is nonempty. Then $\Gamma \setminus \overline{S}$ consists of arcs joining S_1 and S_2.

Assume by contradiction that Γ contains an arc with both extrema on ∂S_i. Without loss of generality we can assume $i = 1$. Then $\Gamma \setminus \overline{S}_2$ is the union of consecutive arcs which are alternatively in S_1 and in $\mathbb{R}^2 \setminus \overline{S}$. By Lemma 3.1, all the arcs of $\Gamma \setminus \overline{S}$ except the two extremal ones are similar, that is, they coincide after a rotation around the center of S_1 (see Figure 2). In particular, at least one of these arcs intersects the complementary of $\mathrm{co}(S)$, contradicting *Step 2.*

Step 5. Let Γ be a connected component of $\partial C_{s,\lambda}$ that intersects both S_1 and S_2. If $\Gamma \cap S_i \neq \emptyset$, for $i = 1, 2$, then $\Gamma \cap S_i$ is an arc of circle of radius $\frac{\lambda}{1-s}$ and $\Gamma \setminus \overline{S}$ consists of two arcs of circle of radius $\frac{\lambda}{s}$, connecting S_1 and S_2.

Let ℓ be the line passing through the centers of S_1 and S_2. Let us consider a coordinate system where the y-axis coincides with ℓ, and S_2 is above S_1. Let Γ_{S_2} be the arc of ∂S_2 contained in $\mathrm{co}(S)$. By going along ∂S_2 in the counterclockwise direction we induce an order in Γ_{S_2}. Similarly, if Γ_{S_1} denotes the arc of ∂S_1 contained in $\mathrm{co}(S)$, we consider the order in Γ_{S_1} induced by going along ∂S_1 clockwise.

Let us order Γ counterclockwise. Since $\Gamma \setminus \overline{S} \neq \emptyset$, we may choose G as the arc in $\Gamma \setminus S$ having greatest intersection point with Γ_{S_2}, with respect to the order of Γ_{S_2}. The arc G intersects Γ_{S_1} at point q, and Γ_{S_2} at a point p. Let γ_{S_2} be the arc of $\Gamma \cap S_2$ starting at p (see Figure 2). Let us observe that if p_1 is the other endpoint of γ_{S_2}, then $p_1 \in \Gamma_{S_2}$ and $p_1 < p$ with respect to the order of Γ_{S_2}. Thus G continues after γ_{S_2} with an arc $G_1 \subset \Gamma \setminus S$ until it intersects S_1 at a point q_1 (see Figure 2, left). Then G_1 enters into S_1 at a point $q_1 < q$. As we observed above, there is an arc $\gamma_{S_1} \subset \Gamma \cap S_1$ that starts at q_1 and exits from S_1 at q_2.

Let G_2 be the arc in $\Gamma \setminus S$ that starts at q_2. We claim that $G_2 = G$. Indeed, notice first that $q_2 \leq q$ by the choice of G. On the other hand, if $q_2 < q$ we could continue following Γ along arcs of circles inside and outside S, until we would reach some point where these arcs intersect each other, giving a contradiction. We thus conclude that $q = q_2$ and hence $G_2 = G$.

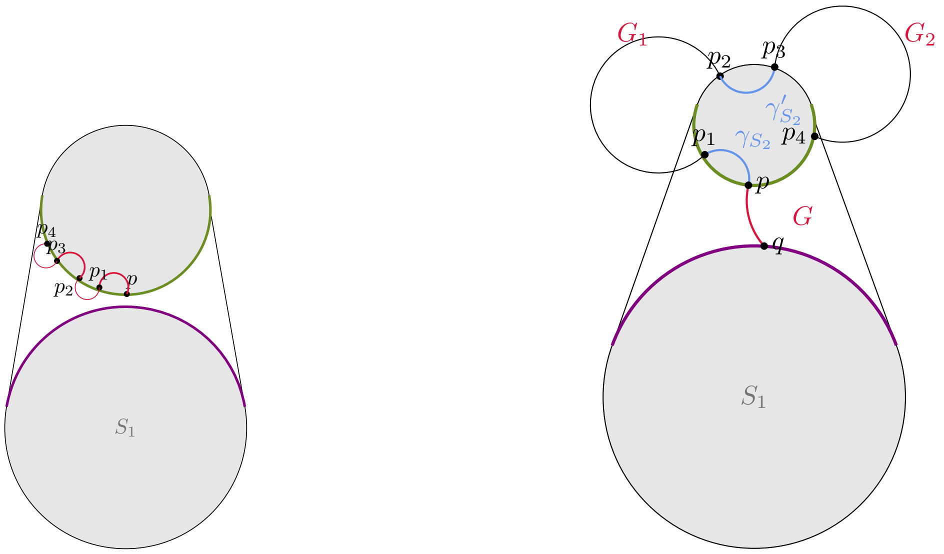

Figure 2: Left: The construction in *Step 4*. Right: Illustration of G, G_1, G_2, γ_{S_2}, γ'_{S_2}, and the points q, p, p_1, p_2 in *Step 5*.

Step 6. Let Γ be a connected component of $\partial C_{s,\lambda}$. Then $\Gamma \setminus \partial S$ is a union circular arcs with angular span strictly less than π.

Let K be a connected component of $\Gamma \setminus \partial S$. Then K is a circular arc of radius $r = r(s, \lambda)$, where $r = \lambda/(1-s)$ if $K \subset S$, and $r = \lambda/s$ if $K \subset \mathbb{R}^2 \setminus \overline{S}$ (if $s = 0$ then K is a segment).

Assume by contradiction that the angular span α of K is greater or equal to π. Then we can modify $C_{s,\lambda}$ and construct a new set with lower energy. Indeed, for $\epsilon > 0$ small enough, we consider a ball B_ϵ of radius $r_\epsilon = (1+\epsilon)r$, containing the endpoints of K. Let $K_\epsilon \subset \partial B_\epsilon$ be the circular arc with the same endpoint as K, and let C_ϵ be the such that $\partial C_\epsilon = (\partial C_{s,\lambda} \setminus K) \cup K_\epsilon$. It is easy to check that $\mathcal{F}_{s,\lambda}(C_\epsilon) < \mathcal{F}_{s,\lambda}(C_{s,\lambda})$, contradicting the minimality of $C_{s,\lambda}$.

Step 7. Let C be a connected component of $C_{s,\lambda}$, then C is simply connected.

If C intersects only S_i then $C = S_i$ be *Step 3*, hence we can assume that C intersects both S_1 and S_2. If C is not simply connected then ∂C contains a closed Jordan curve Γ which bounds a bounded connected component of $\mathbb{R}^2 \setminus C$. By the previous discussion we can write $\Gamma = \cup_{i=1}^4 \Gamma_i$, where Γ_i are circular arcs, Γ_1, Γ_2 have curvature $-(1-s)/\lambda$ and are contained in S_1, S_2 respectively, and Γ_3, Γ_4 have curvature s/λ and are contained in $\mathbb{R}^2 \setminus S$.

Since the curvature κ of Γ is negative on $\Gamma_1 \cup \Gamma_2$ and positive on $\Gamma_3 \cup \Gamma_4$, we have

$$\int_{\Gamma_3 \cup \Gamma_4} \kappa \, d\mathcal{H}^1 = 2\pi - \int_{\Gamma_1 \cup \Gamma_2} \kappa \, d\mathcal{H}^1 \geq 2\pi.$$

On the other hand,

$$\int_{\Gamma_3 \cup \Gamma_4} \kappa \, d\mathcal{H}^1 < 2\pi$$

since by *Step 6* we know that Γ_i have all angular span strictly less than π.

Step 8. Let C be a connected component of $C_{s,\lambda}$ intersecting both S_1 and S_2, then C contains S_1 or S_2. In particular, the set $C_{s,\lambda}$ is connected.

If C contains neither S_1 nor S_2, we can write $\partial C = \cup_{i=1}^4 \Gamma_i$, where Γ_i are circular arcs, Γ_1, Γ_2 have curvature $(1-s)/\lambda$ and are contained in S_1, S_2 respectively, and Γ_3, Γ_4 have curvature $-s/\lambda$ and are contained in $\mathbb{R}^2 \setminus S$. Reasoning as in *Step 7* we then reach a contradiction.

Assume now that $C_{s,\lambda}$ is not connected and let $\tilde{C}$ be a connected component different from C. By the previous discussion, $\tilde{C}$ contains either S_1 or S_2, hence it intersects C, thus giving a contradiction.

Step 9. $C_{s,\lambda}$ is symmetric with respect to ℓ. Moreover, if $r_1 = r_2$, then $C_{s,\lambda}$ is also symmetric with respect to the line ℓ' which is orthogonal to ℓ and has the same distance from S_1 and S_2.

Let $\tilde{C}_{s,\lambda}$ be the set obtained by reflecting $C_{s,\lambda}$ through ℓ, which is still a minimizer of $\mathcal{F}_{s,\lambda}$. Letting $A = C_{s,\lambda} \cap \tilde{C}_{s,\lambda}$, $B = C_{s,\lambda} \cup \tilde{C}_{s,\lambda}$, we have

$$\mathcal{F}_{s,\lambda}(A) + \mathcal{F}_{s,\lambda}(B) = \mathcal{F}_{s,\lambda}(C_{s,\lambda}) + \mathcal{F}_{s,\lambda}(\tilde{C}_{s,\lambda}),$$

which implies that both A and B are minimizers of $\mathcal{F}_{s,\lambda}$. In particular, A and B have boundaries of class $C^{1,1}$, and this is possible only if $C_{s,\lambda} = \tilde{C}_{s,\lambda}$.

The second assertion can be proved analogously by replacing ℓ with ℓ' in the reflection argument.

Step 10. If $C_{s,\lambda}$ is nonempty and different from S_2 then it contains S_1. If $r_1 = r_2$ then $C_{s,\lambda}$ contains S.

Assume by contradiction that $C_{s,\lambda}$ does not contain S_1. Then from the previous steps it follows that $C_{s,\lambda}$ contains S_2 and intersects S_1 in a circular arc. If $r_1 = r_2$ this violates the symmetry of $C_{s,\lambda}$ with respect to ℓ' and gives a contradiction.

Let us consider the case $r_1 > r_2$, and let $\tilde{C}_{s,\lambda}$ (resp. $\tilde{S}_1$) be the sets obtained by reflecting $C_{s,\lambda}$ (resp. S_1) through ℓ'. Let also

$$A = C_{s,\lambda} \cap \left(\tilde{S}_1 \setminus S_2\right) \qquad B = \tilde{C}_{s,\lambda} \cap \left(\tilde{S}_1 \setminus S_2\right)$$

It is easy to check that $B \subset A$ and

$$\mathcal{F}_{s,\lambda}(C_{s,\lambda}) - \mathcal{F}_{s,\lambda}(\tilde{C}_{s,\lambda}) = \frac{1}{\lambda}\left(|A| - |B|\right) > 0\,,$$

contradicting the minimality of $C_{s,\lambda}$.

Step 11. From the previous discussion it follows that either $C_{s,\lambda} \in \{\emptyset, S_1, S\}$, or $C_{s,\lambda}$ is simply connected, contains S_1 and intersects S_2.

□

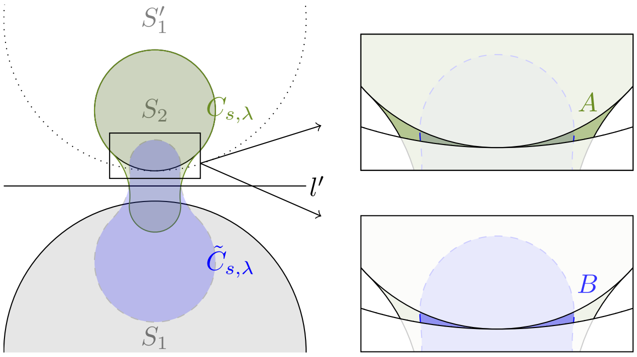

Figure 3: Illustration of *Step 10*.

Corollary 3.3 ([1])**.** *Assume that S_1 and S_2 are two open disjoint balls with equal radius. Let $C_{s,\lambda}$ be a minimizer of $\mathcal{F}_{s,\lambda}$, $s \in [0,1]$. Then if the set $C_{s,\lambda}$ is non-empty, the boundary $\partial C_{s,\lambda}$ is of class $C^{1,1}$. Moreover, $C_{s,\lambda} \cap S_i = S_i$ or $\emptyset$ for any $\lambda > 0$, $s \in [0,1]$ and $i = 1, 2$. In particular, when they exist, the arcs of $\partial C_{s,\lambda} \setminus \overline{S}$ have radius $\frac{\lambda}{s}$ and are tangent to ∂S.*

Example. We give an example of a situation where case b) of Proposition 3.2 is realized. For that, we consider two disjoint balls S_1 and S_2 and assume that they are tangent. Assume also that $r_2 < 1 < r_1$ and take $\lambda = 1$. Then for an appropriate choice of r_1, r_2, the function has level sets that are transversal to S_2, that is, they intersect S_2 but do not contain it. Let $C_\lambda = \{u_\lambda > 0\}$. Since C_λ is a minimizer of the functional $\mathcal{F}_{0,1}(X) = P(X) - |X \cap S|$, it follows that the maximum of the curvature of ∂C_λ is less than 1. However, if $C_\lambda \supseteq S_2$, then the maximum of the curvature is $\frac{1}{r_2} > 1$, is less than 1, contradicting our choice of r_2. If we prove that $C_\lambda \neq S_1$, then C_λ is of the type described in Proposition 3.2 b). For that, it suffices to show that $\mathcal{F}_{0,1}(\mathrm{co}(S)) - \mathcal{F}_{0,1}(S_1) < 0$. Indeed, for $r_2 \ll r_1$, this difference is bounded by

$$\eta = C\frac{r_2^{3/2}}{r_1^{1/2}} - \pi r_2^2,$$

for some constant $C > 0$ independent of r_1, r_2. If we choose $r_1 = \frac{M}{r_2}$ and $M > \frac{C^2}{\pi^2}$, then $\eta = \left(\frac{C}{M^{1/2}} - \pi\right) r_2^2 < 0$.

Definition 3.4. *We call* transversal sets *the sets satisfying condition b) in Proposition 3.2, and we denote them by $T_{s,\lambda}$. If for a given couple of (s, λ) we have two transversal sets, denoted by $T^+_{s,\lambda}, T^-_{s,\lambda}$, with $T^+_{s,\lambda} \subset T^-s, \lambda$, then we say, that $T^+_{s,\lambda}$ is of increasing type and $T^-_{s,\lambda}$ is of decreasing type (see Figure 4).*

Assume that for both combinations $(s_1, \lambda_1), (s_2, \lambda_2)$ with according radii $t_1 = \frac{\lambda_1}{1-s_1} < t_2 = \frac{\lambda_2}{1-s_2}$ we have two transversal sets that we denote by $\left\{T^-_{s_1,\lambda_1}, T^+_{s_1,\lambda_1}\right\}, \left\{T^-_{s_2,\lambda_2}, T^+_{s_2,\lambda_2}\right\}$ respectively.

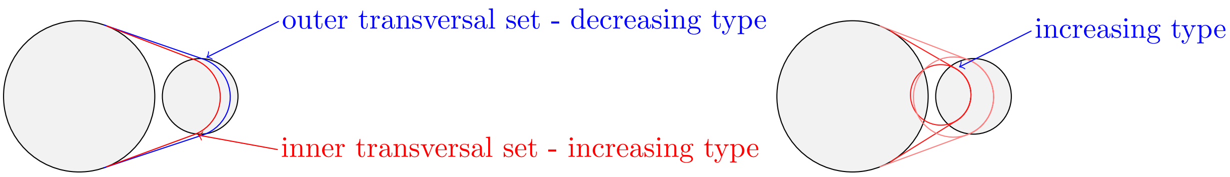

Figure 4: Left: for some values of (s, λ), there are two transversal sets. We call the inner one of increasing type, because for increasing radius (connected to s, λ, these sets decrease, whereas the decreasing sets decrease. Right: For transversal sets of increasing type, by increasing the radius of the inner arc, the transversal set increases.

Moreover, by definition of increasing and decreasing transversal sets, we have $T_i^- \subset T_i^+, i = 1, 2$. Then $T_1^- \subset T_2^-$ and $T_2^+ \subset T_1^+$, meaning that if we increase the radius from t_1 to t_2 the sets of increasing type increase and the sets of decreasing type decrease.

The following Lemma is needed later to state that transversal sets of increasing type cannot be minimizers of $\mathcal{F}_{s,\lambda}$.

Lemma 3.5. *If $\lambda < \mu$, then $C_{s,\mu} \cap S \subset C_{s,\lambda} \cap S$, where $C_{s,\mu}, C_{s,\lambda}$ are minimizers of $\mathcal{F}_{s,\mu}, \mathcal{F}_{s,\lambda}$ respectively.*

Proof. We know from [5], Proposition 3.3.8, that

$$P(C_{s,\lambda} \cup C_{s,\mu}) + P(C_{s,\lambda} \cap C_{s,\mu}) \leq P(C_{s,\lambda}) + P(C_{s,\mu})$$

implying

$$\begin{aligned} \lambda \left(P(C_{s,\lambda} \cup C_{s,\mu}) - P(C_{s,\lambda})\right) &\leq \lambda \left(P(C_{s,\mu}) - P(C_{s,\lambda} \cap C_{s,\mu}\right) \\ &\leq \mu \left(P(C_{s,\mu}) - P(C_{s,\lambda} \cap C_{s,\mu})\right) \ . \end{aligned} \tag{8}$$

The last inequality is strict iff $|P(C_{s,\mu}) - P(C_{s,\lambda} \cap C_{s,\mu})| > 0$.

Because of the optimality of $C_{s,\lambda}$ and $C_{s,\mu}$, we have $\mathcal{F}_{s,\lambda}(C_{s,\lambda}) \leq \mathcal{F}_{s,\lambda}(C_{s,\lambda} \cup C_{s,\mu})$, $\mathcal{F}_{s,\mu}(C_{s,\mu}) \leq \mathcal{F}_{s,\mu}(C_{s,\lambda} \cap C_{s,\mu})$ implying

$$\begin{aligned} &\lambda P\left(C_{s,\lambda}\right) + s\left|C_{s,\lambda} \setminus S\right| - (1-s)\left|C_{s,\lambda} \cap S\right| \\ &\qquad \leq \lambda P\left(C_{s,\lambda} \cup C_{s,\mu}\right) + s\left|(C_{s,\lambda} \cup C_{s,\mu}) \setminus S\right| - (1-s)\left|(C_{s,\lambda} \cup C_{s,\mu}) \cap S\right| \\ &\mu P\left(C_{s,\mu}\right) + s\left|C_{s,\mu} \setminus S\right| - (1-s)\left|C_{s,\mu} \cap S\right| \\ &\qquad \leq \mu P\left(C_{s,\lambda} \cap C_{s,\mu}\right) + s\left|(C_{s,\lambda} \cap C_{s,\mu}) \setminus S\right| - (1-s)\left|(C_{s,\lambda} \cap C_{s,\mu}) \cap S\right| \end{aligned}$$

such that

$$\begin{aligned} &\mu \left(P(C_{s,\mu}) - P(C_{s,\lambda} \cap C_{s,\mu})\right) \\ &\quad \leq \left(|C_{s,\mu} \cap S| - |(C_{s,\lambda} \cap C_{s,\mu}) \cap S|\right) \\ &\quad + s\left(|(C_{s,\lambda} \cap C_{s,\mu}) \setminus S| - |C_{s,\mu} \setminus S| - |C_{s,\mu} \cap S| + |(C_{s,\lambda} \cap C_{s,\mu}) \cap S|\right) \\ &\quad \leq \lambda \left(P(C_{s,\lambda} \cup C_{s,\mu}) - P(C_{s,\lambda})\right) \ . \end{aligned} \tag{9}$$

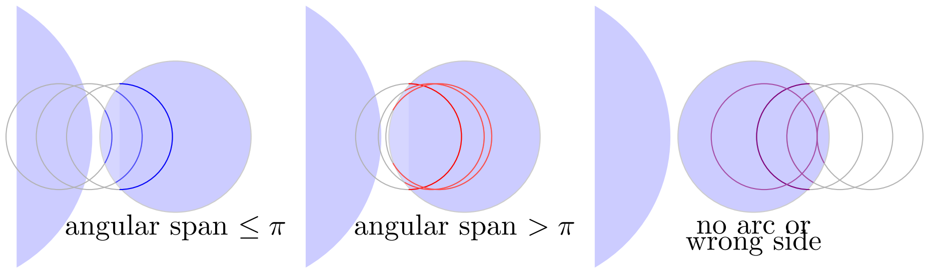

Figure 5: Moving a circle with radius $r < r_2$ from left to right through the circle with radius r_2, we observe the following three cases: a) there is an arc with angular span $\leq \pi$ and the tangents direct to S_1, b) there is an arc with angular span $> \pi$ and the tangents direct to S_1, c) the circle lies inside S_2, or the tangents do not point towards S_1.

(8) and (9) can only hold true if $(P(C_{s,\mu}) - P(C_{s,\lambda} \cap C_{s,\mu}))$ which implies

$$(|C_{s,\mu} \cap S| - |(C_{s,\lambda} \cap C_{s,\mu}) \cap S|) + s\,(|(C_{s,\lambda} \cap C_{s,\mu})| - |C_{s,\mu}|) = 0. \tag{10}$$

This in turn implies that

$$|C_{s,\mu} \cap S| = |(C_{s,\lambda} \cap C_{s,\mu}) \cap S| \qquad |(C_{s,\lambda} \cap C_{s,\mu})| = |C_{s,\mu}|\,.$$

With this we conclude the Lemma. □

Lemma 3.6. *1. For λ, s such that $\frac{\lambda}{1-s} \leq r_2$, there is at most one transversal set.*

2. Assume $s, \mu \leq \lambda$ with $\frac{\lambda}{1-s} \leq r_2$ are such that there exist $T_{s,\lambda}, T_{s,\mu}$ two transversal sets with $T_{s,\mu} \cap S_2 \neq \emptyset$ and $T_{s,\lambda} \cap S_2 \neq \emptyset$. Then $T_{s,\mu} \subset T_{s,\lambda}$.

3. If $\lambda < r_2(1-s)$, then $T_{s,\lambda}$ cannot be a minimzer of $\mathcal{F}_{s,\lambda}$.

4. The sets $\Gamma_{s,\lambda}(S)$ can be transversal sets of decreasing type, equal to S_1 or Close$_{\lambda/s}(S)$.

Proof. 1. Assume s, λ such that $\frac{\lambda}{1-s} \leq r_2$. Set $r_i := \frac{\lambda}{1-s}, r_o := \frac{\lambda}{s}$. Assume that S_1 and S_2 are as in Figure 5, that is, S_1 is on the left side of S_2 and the centers are located at $(-r_1 - r_2 - d, 0)$ and $(0, 0)$ respectively.

Moving a circle with radius $r < r_2$ from left to right through S_1, we observe the following three cases

a) there is an arc with angular span $\leq \pi$ and the tangents direct towards S_1

b) there is an arc with angular span $> \pi$ and the tangents direct towards S_1

c) the circle lies inside S_2, or the tangents do not point towards S_1.

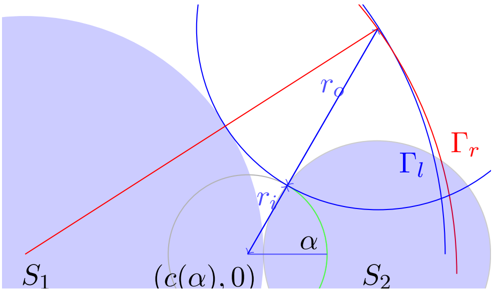

Figure 6: Left: if Γ_r, Γ_l intersect once in the first quadrant, there is one unique transversal set. Right: Case of two transversal sets (inner and outer).

Set α the half angular span of a circle, centered at $(c(\alpha), 0)$ intersecting with ∂S_2 as shown in Figure 6. Explicitly we have $c(\alpha) = -\left(r_i \cos(\alpha) + r_2 \cos\left(\arcsin\left(\frac{r_i}{r_2}\sin(\alpha)\right)\right)\right)$. We can restrict our attention to circles with centers in $(c(\beta), 0)$ for $\beta \in (0, \pi/2)$ such that $c(\beta) \in \left(-(r_2 + r_i), -\sqrt{r_2^2 - r_i^2}\right)$, case a).

To show that there exists maximal one transversal set in the case where $r_i \leq r_2$ we use the following construction: Set

$$\gamma_l(\beta) := (r_1 + r_o)\begin{pmatrix}\cos(\beta)\\ \sin(\beta)\end{pmatrix}$$

$$\gamma_r(\beta) := \begin{pmatrix}c(\alpha) + (r_i + r_o)\cos(\alpha)\\ (r_i + r_o)\sin(\alpha))\end{pmatrix}$$

and $\Gamma_l := \{\gamma_l(\beta), \beta \in (0, \pi/2)\}, \Gamma_r := \{\gamma_r(\alpha), \alpha \in (0, \pi/2)\}$.

Γ_l contain the centers of the circles with radius $\frac{\lambda}{s}$ that are tangential to ∂S_1. Γ_r contains the centers of circles that area tangential to the arcs inside S_2 at the intersection point with ∂S_2.

For a transversal set $\Gamma_{s,\lambda}$, the center of the arc $\partial\Gamma_{s,\lambda} \setminus S$ in the positive y-plane must be an element of $\Gamma_s \cap \Gamma_l$ (condition to be smooth).

Γ_l, Γ_r can be written as to stricly increasing, concave functions, hence they intersect at most once, i.e. there is only one set (β_0, α_0) with $\gamma_l(\beta_0) = \gamma_r(\alpha_0)$. This implies that there is at most one transversal set.

2. Denote by α_μ, β_μ the angular span the connected component of $\partial T_{s,\mu} \cap S_2, \partial T_{s,\mu} \setminus S$, respectively. Moreover denote by p_1, p_2 the intersection points of $\partial T_{s,\mu}$ and ∂S_2. An arc inside S_2 with radius $\frac{\lambda}{1-s}$ intersecting with ∂S_2 at p_1, p_2 has an angular span denoted by α_λ that is smaller than α_μ. Now if we continue smoothly with an arc with radius $\frac{\lambda}{s}$ at p_1, p_2, then this arc will not intersect with ∂S_1. In order to increase the angular span α_λ,

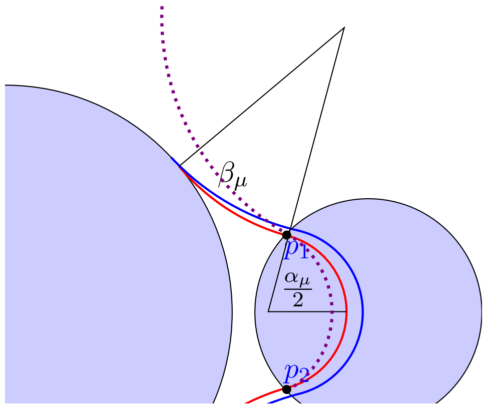

Figure 7: Increasing the radius of the ball touching the points p_1, p_2, the smooth continuation with a circle, does not toucht S_1 anymore. Moving the circle with larger radius to the right, the outer circle touches again S_1.

we have to move the arc inside S_2 outside of $T_{\mu,s}$ until the arc outside of S_2 intersects with ∂S_1. Hence at the end we see that $T_{\mu,s} \cap S \subset T_{\lambda,s} \cap S$.

3. From the previous two items we know that if $\lambda \leq r_2(1-s)$, a transversal set is of increasing type. Lemma 3.5 states that transversal sets of increasing type cannot be a minimizers.

4. Follows from the previous 3 items, together with Lemma 3.5, that states that transversal sets of increasing type cannot be minimizers of $\mathcal{F}_{s,\lambda}$.

□

4 The dual norm of χ_S

In this section we compute the dual norm $\|\chi_S\|_*$. If $g = \chi_E$, then $\|\chi_E\|_* \leq \mu$ if and only if

$$0 \leq \min_{F \subset \mathbb{R}^2} \left\{ P(F) - \frac{1}{\mu} |F \cap E| \right\} = \min_{F \subset \mathbb{R}^2} \mathcal{F}_\mu (F) \ .$$

Let $C_{0,\lambda}$ minimize $\mathcal{F}_{0,\lambda}$, then if $\lambda = \frac{|C_{0,\lambda} \cap S|}{P(C_{0,\lambda})}$, λ is the minimal parameter such that $u = 0$ minimizes (1) and $\|\chi_S\|_* = \frac{|C_{0,\lambda} \cap S|}{P(C_{0,\lambda})}$. Hence, letting $\rho(X) := \frac{|X \cap S|}{P(X)}$ for $X \subset \mathbb{R}^2$, we have

$$\|\chi_S\|_* = \max_{X \subset \mathbb{R}^2} \rho(X) \, . \tag{11}$$

Proposition 4.1. *We have*

$$\|\chi_S\|_* = \max \left\{ \frac{|S_1|}{P(S_1)}, \frac{|S|}{P(co(S))} \right\} \ .$$

For $r \in (0, r_1)$ we let $S(r) := S_1 \cup S_2(r)$, where $S_2(r)$ is a ball of radius r centered at the center of S_2. Before proving the proposition, we need the following Lemma:

Lemma 4.2. *If S_1 maximizes $X \to \frac{|X \cap S(R)|}{P(X)}$ over all $X \subset \mathbb{R}^2$, for some $R > 0$, then it also maximizes $X \to \frac{|X \cap S(r)|}{P(X)}$, for every $0 < r < R$.*

Proof. Note that if X maximizes

$$X \to \frac{|X \cap S(r)|}{P(X)} \tag{12}$$

then X minimizes $\mathcal{F}_{0,\lambda}$ for $\lambda = \frac{|X \cap S(r)|}{P(S(r))}$. Hence by Proposition 3.2 we know that any possible maximizer of (12) must be in

$$\left\{S_1, S(r), co(S(r)), T_\lambda^+(S(r))\right\} .$$

Since by assumption $r \leq r_1$, we can exclude $S(r)$ since

$$\frac{|S_1|}{P(S_1)} = \frac{r_1}{2} \geq \frac{r^2 + r_1^2}{2(r + r_1)} = \frac{|S(r)|}{P(S(r))} .$$

It remains to exclude transversal sets. Assume by contradiction that we can find $X_2 \subset \mathbb{R}^2$ with $X_2 \cap S_2(r) \notin \{\emptyset, S_2(r)\}$, that maximizes (12). Then $\frac{|S_1|}{P(S_1)} \leq \frac{|X_2 \cap S(r)|}{P(X_2)}$.

Since by assumption S_1 maximizes $X \to \frac{|X \cap S(R)|}{P(X)}$, we get $\frac{|X_2 \cap S(R)|}{P(X_2)} \leq \frac{|S_1|}{P(S_1)}$. Both conditions together yield

$$\frac{|X_2 \cap S(R)|}{P(X_2)} \leq \frac{|S_1|}{P(S_1)} \leq \frac{|X_2 \cap S(r)|}{P(X_2)}$$

such that $|X_2 \cap S(R_2)| \leq |X_2 \cap S(r_2)|$ contradicting the assumption $r_2 < R_2$ $(S(r_2) \subset S(R_2))$, hence we conclude the statement. ☐

Now we are ready to prove Proposition 4.1:

Proof. Set $\hat{r}_2$ as the radius larger than zero, such that $\frac{|S_1 \cup S_2(\hat{r}_2)|}{P(co(S(\hat{r}_2)))} = \frac{|S_1|}{P(S_1)}$. It is basic calculus to proof that $\hat{r}_2$ exists and for $r_2 < \hat{r}_2$ $\frac{|S_1 \cup S_2(r_2)|}{P(co(S(r_2)))} > \frac{|S_1|}{P(S_1)}$ and $\frac{|S_1 \cup S_2(r_2)|}{P(co(S(r_2)))} < \frac{|S_1|}{P(S_1)}$ for $r_2 > \hat{r}_2$ (see Figure 8).

Consider the following cases:

- $r_2 > \hat{r}_2$. Set $\lambda := \frac{|S_1 \cup S(r_2)|}{P(co(S(r_2)))}$. Then $\lambda < r_2$ (see Figure 8). This implies that there is no transversal set minimizing $\mathcal{F}_{0,\lambda}$. The only choices for minimizers are S_1 and $co(S)$, since we assume $r_2 > \hat{r}_2$, we have $\mathcal{F}_{0,\lambda}(S_1) > \mathcal{F}_{0,\lambda}(co(S)) = \mathcal{F}_{0,\lambda}(\emptyset) = 0$, hence $co(S)$ is the optimal set and $co(S)$ maximizes (12).

- Case $r_2 = \hat{r}_2$ analog to the previous case, but $\mathcal{F}_{0,\lambda}(S_1) = \mathcal{F}_{0,\lambda}(co(S)) = \mathcal{F}_{0,\lambda}(\emptyset) = 0$, hence $co(S), S_1$ are the optimal sets and $S_1, co(S)$ maximize (12).

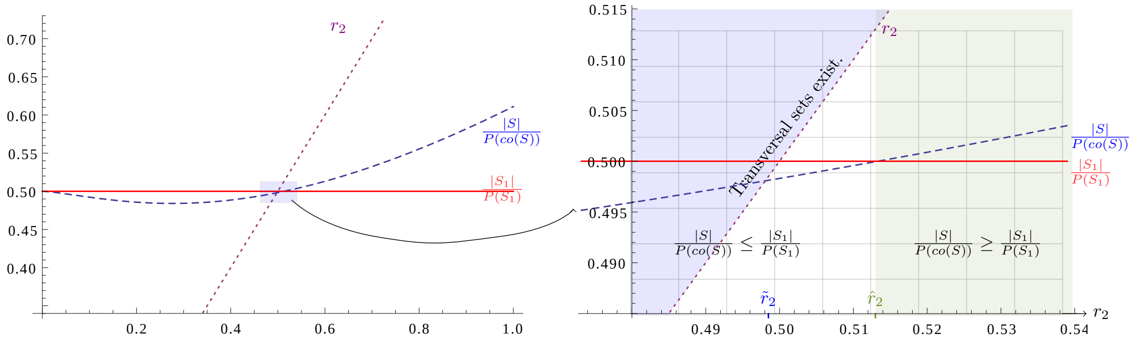

Figure 8: We fix $r_1 = 1$, vary r_2 and look at the ratios of $\frac{S_1}{P(S_1)}$ and $\frac{S}{P(co(S))}$. Solid line $\frac{|S_1|}{P(S_1)}$, dashed line $\frac{|S(r_2)|}{P(co(S(r_2)))}$ for different r_2, dotted line r_2. Left: general situation, right: zoom around $r_2 = 0.5$.

- Case $r_2 < \hat{r}_2$. We can find $\epsilon > 0$ such that for For $r_2 = \hat{r}_2 - \epsilon$, $\frac{|S_1 \cup S_2(r_2)|}{P(co(S(r_2))}< r_2$. Hence also in this case there would be no optimal transversal set. Moreover we have $\frac{|S_1 \cup S_2(r_2)|}{P(co(S(r_2))} < \frac{|S_1|}{P(S_1)}$, hence by Lemma 4.2, we can conclude that for all $r_2 < \hat{r}_2 - \epsilon$, S_1 maximizes $X \to \frac{|X \cap S(r_2)|}{P(X)}$.

Hence we conclude that the only possible choices for an optimal set of (12) are S_1 and $co(S(r_2))$ and conclude the Lemma. $\square$

5 Construction of the minimizers $C_{s,\lambda}$

Construction 5.1. *Let $\lambda > 0$ and $s \in [0, 1]$. Understand $Close_{\frac{\lambda}{0}}(S)$ as $co(S)$ (convex hull of S).*

1. *Set $k := 0$, $X^0_{s,\lambda} := S$, $Y^0_{s,\lambda} := Close_{\frac{\lambda}{s}}(S)$.*
2. *For $k = k + 1$ set*
$$X^k_{s,\lambda} := Open_{\frac{\lambda}{1-s}}\left(Y^{k-1}_{s,\lambda}\right),$$
and
$$Y^k_{s,\lambda} := Close_{\frac{\lambda}{s}}\left(X^k_{s,\lambda} \cap S\right) .$$
3. *Finally define $\Gamma_{s,\lambda}(S) := \bigcap_{k\in\mathbb{N}} Y^k_{s,\lambda}$ $(= \lim_{k\to\infty} Y^k_{s,\lambda})$.*

Remark 5.2. The sets $X^k_{s,\lambda}$ and $Y^k_{s,\lambda}$ have the following properties:

i) $X^0_{s,\lambda} \subset Y^0_{s,\lambda}$ and $Y^0_{s,\lambda} \cap S \subset X^0_{s,\lambda}$ such that $X^1_{s,\lambda} = Open_{\frac{\lambda}{1-s}}\left(Y^0_{s,\lambda} \cap S\right) \subset Open_{\frac{\lambda}{1-s}}\left(X^0_{s,\lambda}\right) \subset X^0_{s,\lambda}$. Consequently $Y^0_{s,\lambda} \subset Y^1_{s,\lambda}$ and so on. In general we have
$$Y^{k+1}_{s,\lambda} \subset Y^k_{s,\lambda} \quad \text{and} \quad X^{k+1}_{s,\lambda} \subset X^k_{s,\lambda} .$$

ii) Due to the properties of the opening and the closing operators (see Lemma 2.2) we have for the curvature κ:

$$\kappa(\partial X^k_{s,\lambda}) \leq \frac{1-s}{\lambda} \qquad \text{and} \qquad \kappa(\partial Y^k_{s,\lambda}) \geq -\frac{s}{\lambda}\ .$$

iii) If $Y^1_{s,\lambda} = X^1_{s,\lambda}$, then $Y^k_{s,\lambda} = Y^1_{s,\lambda}$ for all $k > 1$ such that $\Gamma_{s,\lambda}(S) = Y^1_{s,\lambda}$.

iv) In the case where $X^k_{s,\lambda} \neq X^{k+1}_{s,\lambda} = \mathit{Open}_{\frac{\lambda}{1-s}}\left(\mathit{Close}_{\frac{\lambda}{s}}\left(X^k_{s,\lambda} \cap S\right)\right) = \mathit{Open}_{\frac{\lambda}{1-s}}\left(Y^k_{s,\lambda}\right)$, there exists a part in $\partial Y^k_{s,\lambda}$ with curvature larger than $\frac{1-s}{\lambda}$. Applying another opening to $Y^k_{s,\lambda}$ we replace this part, but then $\partial \mathit{Open}_{\frac{\lambda}{1-s}}\left(Y^k_{s,\lambda}\right) \setminus S$ might have parts with curvature different from $-\frac{s}{\lambda}$.

v) For every k we have $\mathit{Open}_{\frac{\lambda}{1-s}}(S) \subset X^k_{s,\lambda}, Y^k_{s,\lambda} \subset \mathit{Close}_{\frac{\lambda}{1-s}}(S)$.

Remark 5.3. The sets $\Gamma_{s,\lambda}(S)$ have the following properties:

a) $\mathit{Open}_{\frac{\lambda}{1-s}}(\Gamma_{s,\lambda}(S)) \cap S = \Gamma_{s,\lambda}(S) \cap S$.

b) $\mathit{Close}_{\frac{\lambda}{s}}(\Gamma_{s,\lambda}(S)) = \Gamma_{s,\lambda}(S)$.

c) $\partial\Gamma_{s,\lambda}(S) \cap S$ has curvature $\frac{1-s}{\lambda}$, and $\partial\Gamma_{s,\lambda}(S) \setminus S$ has curvature $-\frac{s}{\lambda}$.

d) $\partial\Gamma_{s,\lambda}(S)$ is of class $\mathcal{C}^{1,1}$ and $-\frac{s}{\lambda} \leq \kappa\left(\partial\Gamma_{s,\lambda}(S)\right) \leq \frac{1-s}{\lambda}$.

e) If $\frac{\lambda}{1-s} \leq r_2 \leq r_1$ then $\Gamma_{s,\lambda}(S) = \mathit{Close}_{\frac{\lambda}{s}}(S)$.

f) For $r_2 < \frac{\lambda}{1-s} \leq r_1$ then $S_1 \subset \Gamma_{s,\lambda}(S) \subsetneq \mathit{Close}_{\frac{\lambda}{s}}(S)$.

g) For $\frac{\lambda}{1-s} > r_1$, $\Gamma_{s,\lambda}(S) = \emptyset$.

We now give an explicit characterization of the solutions of (1).

Remark 5.4. By Lemma 2.6, we may assume that

$$P(S) > P(\mathrm{co}(S)), \tag{13}$$

otherwise the solution corresponding to S is described by the sum of the solutions corresponding to S_1 and S_2, that is, both sets do not interact. Moreover, by Lemma 4.1 and Proposition 2.4 it is enough to consider

$$\lambda \leq \max\left\{\frac{|S_1|}{P(S_1)}, \frac{|S|}{P(\mathrm{co}(S))}\right\},$$

otherwise the solution of (1) is equal to zero.

We start by comparing the energies of S_1, S_2, S and $\emptyset$.

Lemma 5.5. *Let $\lambda > 0$. We have*

$$\min\left\{0, \mathcal{F}_{s,\lambda}(S), \mathcal{F}_{s,\lambda}(S_1), \mathcal{F}_{s,\lambda}(S_2)\right\} = \begin{cases} \mathcal{F}_{s,\lambda}(S) & \frac{P(S_2)}{|S_2|} \le \frac{1-s}{\lambda}, \\ \mathcal{F}_{s,\lambda}(S_1) & \frac{P(S_1)}{|S_1|} \le \frac{1-s}{\lambda} \le \frac{P(S_2)}{|S_2|}, \\ 0 & \frac{1-s}{\lambda} \le \frac{P(S_1)}{|S_1|}. \end{cases}$$

Proof. Observe that, for $i = 1, 2$,

$$\mathcal{F}_{s,\lambda}(S_i) = P(S_i) - \frac{1-s}{\lambda}|S_i| \le 0 \quad \text{if and only if} \quad \frac{1-s}{\lambda} \ge \frac{P(S_i)}{|S_i|}.$$

If $\frac{1-s}{\lambda} \ge \frac{P(S_2)}{|S_2|}$, then $\mathcal{F}_{s,\lambda}(S) = \mathcal{F}_{s,\lambda}(S_1) + \mathcal{F}_{s,\lambda}(S_2) \le \min\{0, \mathcal{F}_{s,\lambda}(S_1), \mathcal{F}_{s,\lambda}(S_2)\}$. If $\frac{P(S_1)}{|S_1|} \le \frac{1-s}{\lambda} < \frac{P(S_2)}{|S_2|}$, then $\mathcal{F}_{s,\lambda}(S_2) > 0$ and $\mathcal{F}_{s,\lambda}(S) = \mathcal{F}_{s,\lambda}(S_1) + \mathcal{F}_{s,\lambda}(S_2) > \mathcal{F}_{s,\lambda}(S_1)$. In case $\frac{1-s}{\lambda} < \frac{P(S_1)}{|S_1|}$, $\mathcal{F}_{s,\lambda}(S_1) > 0$, $\mathcal{F}_{s,\lambda}(S_2) > 0$ and $\min\{0, \mathcal{F}_{s,\lambda}(S), \mathcal{F}_{s,\lambda}(S_1), \mathcal{F}_{s,\lambda}(S_2)\} = 0$. ☐

We now define special values of λ which will be useful in order to classify the minimizers of $\mathcal{F}_{s,\lambda}$.

Proposition 5.6. *Assume that* (13) *holds. Let $R_c(S)$ be the minimal radius r such that $Close_r(S)$ is connected.*

(a) There is a unique value $R_1 \in [R_c(S), \infty)$ such that

$$P\left(Close_{R_1}(S)\right) + \frac{1}{R_1}\left|Close_{R_1}(S) \setminus S\right| = P(S). \tag{14}$$

Let $s_2(\lambda) = 1 - \lambda\frac{P(S_2)}{|S_2|}$ and λ_1 be given by $\frac{\lambda_1}{s_2(\lambda_1)} = R_1$. Then $\lambda_1 := \frac{R_1|S_2|}{R_1 P(S_2) + |S_2|} \in \left[0, \frac{|S_2|}{P(S_2)}\right]$ and

$$\mathcal{F}_{s_2(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_2(\lambda)}}(S)\right) > \mathcal{F}_{s_2(\lambda),\lambda}(S) \qquad \text{(resp. } =, <\text{)} \tag{15}$$

for any $\lambda < \lambda_1$ (resp $=$, $>$). The value $\lambda_1 = 0$ if and only if $R_1 = 0$, and this happens if and only if $R_c(S) = 0$, in other words if the two balls touch eachother. If $R_c(S) > 0$, then $R_1 > R_c(S)$.

(b) i) If $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$, there is a unique value $R_2 \in [R_c(S), \infty)$ such that

$$P\left(Close_{R_2}(S)\right) + \frac{1}{R_2}\left|Close_{R_2}(S) \setminus S\right| = P(S_1)\frac{|S|}{|S_1|}. \tag{16}$$

Let $s_1(\lambda) := 1 - \lambda\frac{P(S_1)}{|S_1|}$ and $\lambda_2 := \frac{R_2|S_1|}{R_2 P(S_1) + |S_1|} \in \left[\lambda_1, \frac{|S_1|}{P(S_1)}\right]$. Then $\frac{\lambda_2}{s_1(\lambda_2)} = R_2$ and

$$\mathcal{F}_{s_1(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_1(\lambda)}}(S)\right) > \mathcal{F}_{s_1(\lambda),\lambda}(S_1) \qquad \text{(resp. } =, <\text{)} \tag{17}$$

for any $\lambda < \lambda_2$ (resp $=$, $>$). We have that $R_1 = R_2$ if and only if $\frac{P(S_1)}{|S_1|} = \frac{P(S_2)}{|S_2|}$ if and only if $\lambda_1 = \lambda_2$. And $R_2 = 0$ (in that case also $R_1 = 0$ and $\lambda_1 = \lambda_2 = 0$) if and only if $R_c(S) = 0$.

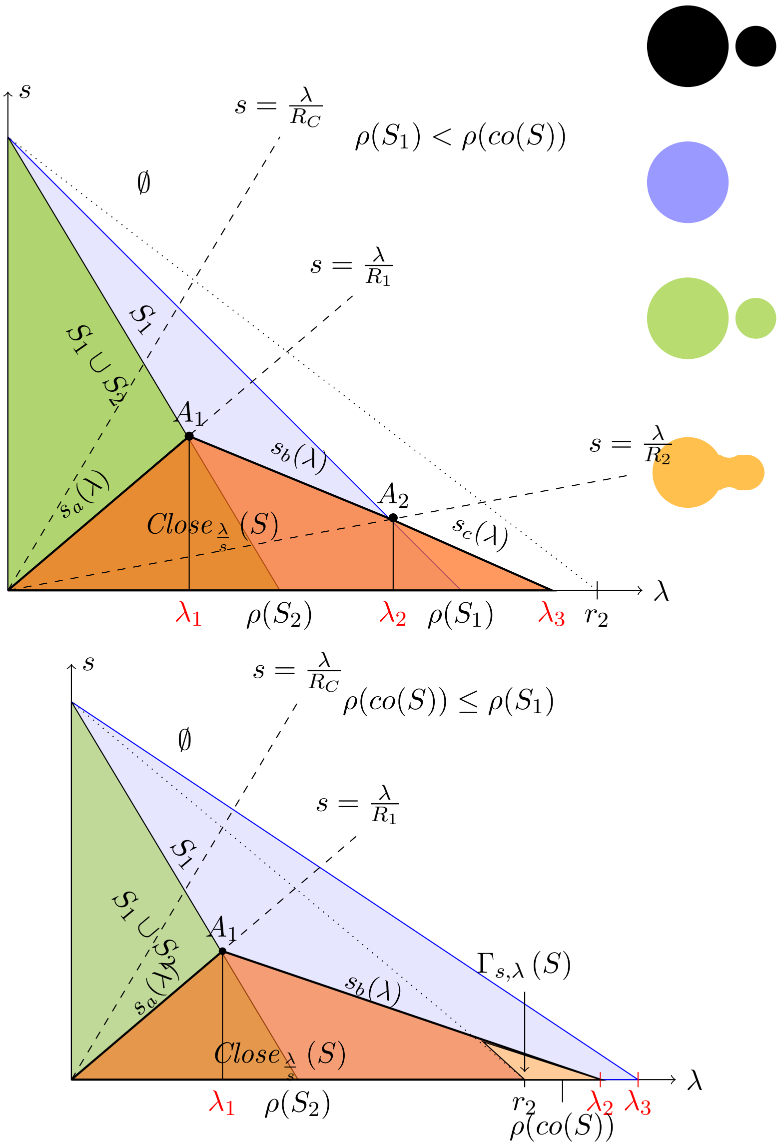

Figure 9: Illustration of Theorem 5.8. For any point (λ, s) this diagram shows which one of the four sets $S_1, S, \Gamma_{s,\lambda}(S), \emptyset$ has the minimal $\mathcal{F}_{s,\lambda}$ value. For $(\lambda, s) = (\lambda, 1-\lambda\frac{P(S_1)}{|S_1|}), (\lambda, 1-\lambda\frac{P(S_2)}{|S_2|})$, the minima are not unique. The dotted line indicates the values of (s, λ) such that $\frac{\lambda}{1-s} = r_2$, hence there are no transversal minimizers on the left of the dotted line.

ii) If $\frac{|S|}{P(\mathrm{co}(S))} \leq \frac{|S_1|}{P(S_1)}$ *set* λ_2 *as the solution of* $\lambda_2 = \frac{|\Gamma_{0,\lambda_2}(S) \cap S_2|}{P(\Gamma_{0,\lambda_2}(S)) - P(S_1)}$.
If $\frac{|S_2|}{P(co(S)) - P(S_1)} \leq r_2$, *then* $\lambda_2 = \frac{|S_2|}{P(co(S)) - P(S_1)}$.

(c) Set $\lambda_3 := \max\left\{\frac{|S_1|}{P(S_1)}, \frac{|S|}{P(\mathrm{co}(S))}\right\}$.

Remark 5.7. Observe that when $\frac{|S_1|}{P(S_1)} = \frac{|S|}{P(\mathrm{co}(S))}$, $\lambda_2 = \frac{|S_1|}{P(S_1)} = \lambda_3$.

Now we are ready to describe the minimizers of $\mathcal{F}_{s,\lambda}$. For simplicity, from now on we denote by $C_{s,\lambda}$ the largest minimizer of $\mathcal{F}_{s,\lambda}$ (see Proposition 2.3).

Theorem 5.8. *Assume that* (13) *holds. Let* $\lambda_1, \lambda_2, \lambda_3$ *be as in Proposition 5.6. Then the sets* $C_{s,\lambda}$ *are given by:*

(a) Let $\lambda \in [0, \lambda_1]$. *There is a value* $0 < s_a(\lambda) \leq 1 - \lambda \frac{P(S_2)}{|S_2|}$ *such that*

$$C_{s,\lambda} = \begin{cases} \mathit{Close}_{\frac{\lambda}{s}}(S) & 0 \leq s \leq s_a(\lambda) \\ S & s_a(\lambda) < s \leq 1 - \lambda \frac{P(S_2)}{|S_2|} \\ S_1 & 1 - \lambda \frac{P(S_2)}{|S_2|} < s \leq 1 - \lambda \frac{P(S_1)}{|S_1|} \\ \emptyset & 1 - \lambda \frac{P(S_1)}{|S_1|} < s. \end{cases}$$

The third interval is empty in the case $\frac{|S_1|}{P(S_1)} = \frac{|S_2|}{P(S_2)}$.

(b) Let $\lambda \in (\lambda_1, \lambda_2]$. *There is a value* $1 - \lambda \frac{|S_2|}{P(S_2)} < s_b(\lambda) \leq 1 - \lambda \frac{P(S_1)}{|S_1|}$ *such that*

$$C_{s,\lambda} = \begin{cases} \Gamma_{s,\lambda}(S) & 0 \leq s \leq s_b(\lambda) \\ S_1 & s_b(\lambda) < s \leq 1 - \lambda \frac{P(S_1)}{|S_1|} \\ \emptyset & 1 - \lambda \frac{P(S_1)}{|S_1|} < s\,, \end{cases}$$

and $\Gamma_{s,\lambda}(S) = \mathit{Close}_{\frac{\lambda}{s}}(S)$ *as long as* $\frac{\lambda}{1-s} \leq r_2$.

(c) Let $\lambda \in (\lambda_2, \lambda_3]$.

(c1) If $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$, *then there is a value* $s_c(\lambda) > 1 - \lambda \frac{P(S_1)}{|S_1|}$ *such that*

$$C_{s,\lambda} = \begin{cases} \mathit{Close}_{\frac{\lambda}{s}}(S) & 0 \leq s \leq s_c(\lambda) \\ \emptyset & \text{else.} \end{cases}$$

(c2) If $\frac{|S|}{P(\mathrm{co}(S))} \leq \frac{|S_1|}{P(S_1)}$, *then*

$$C_{s,\lambda} = \begin{cases} S_1 & 0 \leq s \leq 1 - \lambda \frac{P(S_1)}{|S_1|} \\ \emptyset & \text{else.} \end{cases}$$

(d) For $\lambda > \lambda_3$ $C_{s,\lambda} = \emptyset$.

Figure 11 shows solutions of (1) for different λ, when S is the union of two balls with radii $r_1 = 1.2, r_2 = 1$ and distance $d = 0.05$ (case $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(co(S))}$).

In order to prove Proposition 5.6 and Theorem 5.8 we need the following Lemmas:

Lemma 5.9. *Let $0 < R < r$ and define*

$$G_r(h) := \int_{-R}^{R} \left(\sqrt{1 + h'(x)^2} - \frac{1}{r} h(x) \right) \, dx. \tag{18}$$

The function that represents an arc of a circle with radius r (angular span smaller than π) from $-R$ to R, minimizes $G_r(h)$ under all functions h with $h(-R) = 0, h(R) = 0$.

Proof. See [7, Lemma 4.29]. □

Lemma 5.10. *Let $R_c(S)$ be the minimal radius r such that $Close_r(S)$ is connected. Then for $s \in [0,1], 0 < \mu < \lambda$, such that $R_c(S) \leq \frac{\mu}{s}$,*

$$\mathcal{F}_{s,\lambda} \left(Close_{\frac{\lambda}{s}}(S) \right) \leq \mathcal{F}_{s,\lambda} \left(Close_{\frac{\mu}{s}}(S) \right) .$$

Proof. Let us take the x-axis as the axis joining the centers of the two circles. Then S is symmetric with respect to the x-axis. Then the upper parts of $\partial \left(Close_{\frac{\lambda}{s}}(S) \right)$ and $\partial \left(Close_{\frac{\mu}{s}}(S) \right)$ are representable as functions $f, g : [a,b] \to \mathbb{R}$, such that

$$\frac{1}{2} \mathcal{F}_{s,\lambda} \left(Close_{\frac{\lambda}{s}}(S) \right) = \int_{[a,b]} \sqrt{1 + (f')^2} + \frac{s}{\lambda} \left(\int_{[a,b]} f - \frac{1}{2} |S| \right)$$

$$\frac{1}{2} \mathcal{F}_{s,\lambda} \left(Close_{\frac{\mu}{s}}(S) \right) = \int_{[a,b]} \sqrt{1 + (g')^2} + \frac{s}{\lambda} \left(\int_{[a,b]} g - \frac{1}{2} |S| \right)$$

Let P, Q be the points in the positive y-plane where $\partial Close_{\frac{\lambda}{s}}(S)$ intersects with S_1 and S_2. Define $h := [a', b'] \to \mathbb{R}$ as the affine function from $P = (a', f(a'))$ to $Q = (b', f(b'))$ (see Figure 10). Set $\tilde{f} := [a', b'] \to \mathbb{R}, \tilde{f} = h - f$ and $\tilde{g} := [a', b'] \to \mathbb{R}, \tilde{g} = h - g$, then $\tilde{g}(a') = \tilde{f}(a') = 0$, $\tilde{g}(b') = \tilde{f}(b') = 0$, ${g'}^2 = (\tilde{g}')^2$ and ${f'}^2 = (\tilde{f}')^2$. Note that $\tilde{f}$ is an arc of circle with radius $\frac{\lambda}{s}$.

Using functional G_r as in (18) with $r = \frac{\lambda}{s}$, replacing the domain of integration $[-R, R]$ by $[a', b']$, and Lemma 5.9 (see also [7, Lemma 4.29]) we have $G_{\frac{\lambda}{s}}(\tilde{f}) \leq G_{\frac{\lambda}{s}}(\tilde{g})$. Hence

$$\frac{1}{2} \mathcal{F}_{s,\lambda} \left(Close_{\frac{\lambda}{s}}(S) \right) - \frac{1}{2} \mathcal{F}_{s,\lambda} \left(Close_{\frac{\mu}{s}}(S) \right) = G_{\frac{\lambda}{s}} \left(\tilde{f} \right) - G_{\frac{\lambda}{s}} \left(\tilde{g} \right) \leq 0.$$

□

Next we show some properties of the function $(s, \lambda) \to \mathcal{F}_{s,\lambda} \left(Close_{\frac{\lambda}{s}}(S) \right)$.

Lemma 5.11. *The function $(s, \lambda) \to \mathcal{F}_{s,\lambda} \left(Close_{\frac{\lambda}{s}}(S) \right)$ satisfies the following properties:*

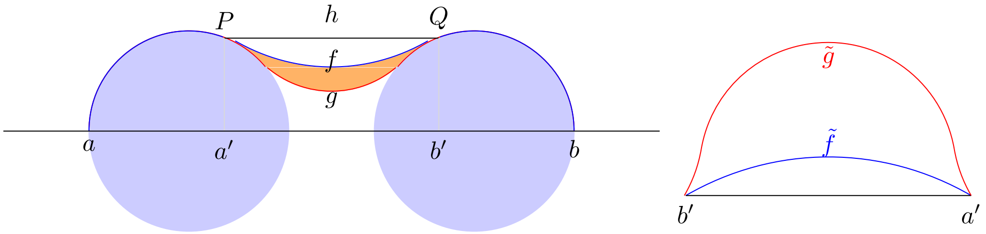

Figure 10: The geometric configuration of the proof of Lemma 5.10.

(i) Let $\kappa > 0$ *and* $s_\kappa(\lambda) := 1 - \frac{\lambda}{\kappa}$, $\lambda > 0$. *Then the mapping* $\lambda \to \mathcal{F}_{s_\kappa(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_\kappa(\lambda)}}(S)\right)$ *is strictly decreasing and continuous as long as* $\frac{\lambda}{s_\kappa(\lambda)} \geq R_c(S)$, *i.e., as long as the set* $Close_{\frac{\lambda}{s_\kappa(\lambda)}}(S)$ *is connected.*

(ii) For $\lambda > 0$, *the mapping* $s \to \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right)$ *is continuous and strictly increasing for* $s \in [0, \frac{\lambda}{R_c(S)}]$, *i.e. as long as the set* $Close_{\frac{\lambda}{s}}(S)$ *is connected.*

(iii) For $r \in [R_c(S), \infty]$ *the functions*

$$r \to P(Close_r(S)) + \frac{1}{r}|Close_r(S)|$$
$$r \to P(Close_r(S)) + \frac{1}{r}|Close_r(S) \setminus S|$$

are continuous and strictly decreasing in r.

Proof. (i) Let $\lambda_1 < \lambda_2$ and set $s_i := 1 - \frac{\lambda_i}{\kappa}, i = 1, 2$. Then $s_2 < s_1$ and $\frac{\lambda_1}{s_1} < \frac{\lambda_2}{s_2}$. Assume that $\frac{\lambda_1}{s_1} \geq R_c(S)$. By Lemma 5.10 we have that

$$\begin{aligned}\mathcal{F}_{s_2,\lambda_2}\left(Close_{\frac{\lambda_2}{s_2}}(S)\right) &\leq \mathcal{F}_{s_2,\lambda_2}\left(Close_{\frac{\lambda_1}{s_1}}(S)\right)\\ &= P\left(Close_{\frac{\lambda_1}{s_1}}(S)\right) - \frac{1}{\kappa}\left|Close_{\frac{\lambda_1}{s_1}}(S) \cap S\right| + \frac{s_2}{\lambda_2}\left|Close_{\frac{\lambda_1}{s_1}}(S) \setminus S\right|\\ &< \mathcal{F}_{s_1,\lambda_1}\left(Close_{\frac{\lambda_1}{s_1}}(S)\right).\end{aligned}$$

Hence the mapping $\lambda \to \mathcal{F}_{s_\kappa(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_\kappa(\lambda)}}(S)\right)$ is strictly decreasing as long as $\frac{\lambda}{s_\kappa(\lambda)} \geq R_c(S)$. The continuity follows from the continuity of the involved functions.

(ii)–(iii) Let $\lambda > 0$. The continuity of the function $s \in [0,1] \to \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right)$ follows from the continuity of the involved functions. Assume $0 \le s_1 < s_2 \le 1$ and λ such that $Close_{\frac{\lambda}{s_1}}(S), Close_{\frac{\lambda}{s_2}}(S)$ are connected. Since $R_c(S) \le \frac{\lambda}{s_2} < \frac{\lambda}{s_1}$, Lemma 5.10 gives

$$\mathcal{F}_{s_1,\lambda}\left(Close_{\frac{\lambda}{s_1}}(S)\right) \le \mathcal{F}_{s_1,\lambda}\left(Close_{\frac{\lambda}{s_2}}(S)\right), \tag{19}$$

hence

$$\begin{aligned}
\mathcal{F}_{s_1,\lambda}\left(Close_{\frac{\lambda}{s_1}}(S)\right) &= P\left(Close_{\frac{\lambda}{s_1}}(S)\right) + \frac{s_1}{\lambda}\left|Close_{\frac{\lambda}{s_1}}(S)\right| - \frac{1}{\lambda}|S| \\
&\le P\left(Close_{\frac{\lambda}{s_2}}(S)\right) + \frac{s_1}{\lambda}\left|Close_{\frac{\lambda}{s_2}}(S)\right| - \frac{1}{\lambda}|S| \\
&< P\left(Close_{\frac{\lambda}{s_2}}(S)\right) + \frac{s_2}{\lambda}\left|Close_{\frac{\lambda}{s_2}}(S)\right| - \frac{1}{\lambda}|S| \\
&= \mathcal{F}_{s_2,\lambda}\left(Close_{\frac{\lambda}{s_2}}(S)\right).
\end{aligned}$$

This proves that the mapping $s \to \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right)$ is strictly increasing on the interval $\left[0, \min\left\{1, \frac{\lambda}{R_c(S)}\right\}\right]$. Adding $\frac{1}{\lambda}|S|$ to both sides of the inequality and setting $r_1 := \frac{\lambda}{s_1}, r_2 := \frac{\lambda}{s_2}$ we have that $0 < r_2 < r_1$ and

$$P\left(Close_{r_1}(S)\right) + \frac{1}{r_1}\left|Close_{r_1}(S)\right| < P\left(Close_{r_2}(S)\right) + \frac{1}{r_2}\left|Close_{r_2}(S)\right|.$$

Hence $P\left(Close_r(S)\right) + \frac{1}{r}\left|Close_r(S)\right|$ is strictly decreasing in r for $r \in [R_c(S), \infty]$ if $R_c(S) > 0$, or in $(R_c(S), \infty]$ if $R_c(S) = 0$. We understand that the function is $+\infty$ when $r = R_c(S) = 0$.

Writing again (19) as

$$\begin{aligned}
& P\left(Close_{\frac{\lambda}{s_1}}(S)\right) + \frac{s_1}{\lambda}\left|Close_{\frac{\lambda}{s_1}}(S) \setminus S\right| - \frac{1-s_1}{\lambda}|S| \\
\le\ & P\left(Close_{\frac{\lambda}{s_2}}(S)\right) + \frac{s_1}{\lambda}\left|Close_{\frac{\lambda}{s_2}}(S) \setminus S\right| - \frac{1-s_1}{\lambda}|S| \\
<\ & P\left(Close_{\frac{\lambda}{s_2}}(S)\right) + \frac{s_2}{\lambda}\left|Close_{\frac{\lambda}{s_2}}(S) \setminus S\right| - \frac{1-s_1}{\lambda}|S|,
\end{aligned}$$

setting $r_1 := \frac{\lambda}{s_1}, r_2 := \frac{\lambda}{s_2}$ (notice that these values are not 0), and adding $\frac{1-s_1}{\lambda}|S|$ to both sides of the inequality above, we get that $P\left(Close_r(S)\right) + \frac{1}{r}\left|Close_r(S) \setminus S\right|$ is strictly decreasing in r for $r \in [R_c(S), \infty]$ if $R_c(S) > 0$, or in $(R_c(S), \infty]$ if $R_c(S) = 0$. Notice that the function is also continuous in that range.

It remains to consider the case $R_c(S) = 0$ and to prove that

$$P\left(\mathit{Close}_r\left(S\right)\right) + \frac{1}{r}\left|\mathit{Close}_r\left(S\right) \setminus S\right|$$

is continuous at $r = 0$. This follows if we prove that

$$\frac{1}{r}\left|\mathit{Close}_r\left(S\right) \setminus S\right| \to 0+ \qquad \text{as } r \to 0+. \tag{20}$$

To prove (20), let us estimate the area of $\mathit{Close}_r\left(S\right) \setminus S$. This set is contained in a triangle whose basis has length $\leq 2r$ and whose height is less than $\sqrt{r}\max(\sqrt{2R_1 - r}, \sqrt{2R_2 - r}) = O(\sqrt{r})$. Hence $\left|\mathit{Close}_r\left(S\right) \setminus S\right| = O(r^{3/2})$ and (20) holds.

□

Lemma 5.12. *Assume that* (13) *holds and that* $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(co(S))}$. *Then* $C_{s,\lambda} \cap S_2 \in \{\emptyset, S_2\}$, *i.e.,* $C_{s,\lambda}$ *cannot be a transversal set.*

Proof. From (11) and Proposition 4.1 we obtain that

$$\lambda_3 := \frac{|S|}{P(co(S))} = \max_{X \subset \mathbb{R}^2} \rho(X)\,. \tag{21}$$

We claim that $\lambda_3 \leq r_2$. Indeed, if $\lambda_3 > r_2$ then by Lemma 5.9 we have that the set $co(S)$ cannot be a minimizer of $\mathcal{F}_{0,\lambda_3}$, that is, there exists $X \subset \mathbb{R}^2$ such that

$$\mathcal{F}_{0,\lambda_3}(X) < \mathcal{F}_{0,\lambda_3}(co(S)) = 0\,.$$

It then follows $\rho(X) > \lambda_3$, contradicting (21). We thus proved that $\lambda_3 \leq r_2$.

We now claim that, for $s > 1 - \frac{\lambda}{\lambda_3}$ we necessarily have $C_{s,\lambda} = \emptyset$. Indeed, it is enough to show that the empty set is a minimizer of $\mathcal{F}_{s,\lambda}$ for $s = 1 - \frac{\lambda}{\lambda_3}$, that is, $\mathcal{F}_{s,\lambda}(C_{s,\lambda}) = 0$. Since $\frac{1-s}{\lambda} = \frac{1}{\lambda_3} \geq \frac{1}{r_2}$, from Proposition 3.2 it follows that $C_{s,\lambda} \in \left\{S_1, S, \emptyset, \mathit{Close}_{\frac{s}{\lambda}}\left(S\right)\right\}$. From our assumptions it directly follows that $\mathcal{F}_{s,\lambda}(S_1) > 0$ and $\mathcal{F}_{s,\lambda}(S) > 0$, hence it remains to show that $\mathcal{F}_{s,\lambda}(\mathit{Close}_{\frac{s}{\lambda}}\left(S\right)) \geq 0$. By Lemma 5.11 (i) we know that

$$\mathcal{F}_{s,\lambda}(\mathit{Close}_{\frac{s}{\lambda}}\left(S\right)) = \mathcal{F}_{1-\frac{\lambda}{\lambda_3},\lambda}(\mathit{Close}_{\frac{1}{\lambda}-\frac{1}{\lambda_3}}(S)) \geq \mathcal{F}_{0,\lambda_3}(co(S)) = 0\,,$$

which proves our claim.

In particular, if $C_{s,\lambda} \neq \emptyset$ it follows that

$$s \leq 1 - \frac{\lambda}{\lambda_3} \leq 1 - \frac{\lambda}{r_2}\,,$$

i.e., $\frac{1-s}{\lambda} \geq \frac{1}{r_2}$, hence $C_{s,\lambda}$ cannot be a transversal set (again by Proposition 3.2 b)). □

Proof of Proposition 5.6.

(a) Assume first that $R_c(S) > 0$. In that case, because of the convexity of S_1, S_2, $P(S) < P(\mathit{Close}_{R_c(S)}(S))$, and we have

$$P(S) < P\left(\mathit{Close}_{R_c(S)}(S)\right) + \frac{1}{R_c(S)}\left|\mathit{Close}_{R_c(S)}(S) \setminus S\right| .$$

On the other hand

$$\lim_{r\to\infty} P(\mathit{Close}_r(S)) + \frac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right| = P(co(S)) < P(S) . \tag{22}$$

Hence, by Lemma 5.11.(iii), (14) has a unique solution in $R_1 \in (R_c(S), \infty)$.

Assume now that $R_c(S) = 0$. Then we have $\mathit{Close}_{R_c(S)}(S) = S$, moreover, since $f(r) := P(\mathit{Close}_r(S)) + \frac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right|$ is a continuous and decreasing function in $[R_c(S), \infty)$ by Lemma 5.11.(iii), we have

$$\lim_{r\to 0+} P(\mathit{Close}_r(S)) + \frac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right| \geq \lim_{r\to 0+} P(\mathit{Close}_r(S)) = P(S).$$

On the other hand, we also have (22). Thus, $R_1 = 0 \in [0, \infty)$ satisfies (14). This value is unique since $r \to P(\mathit{Close}_r(S)) + \frac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right|$ is strictly decreasing.

To prove (15), let us observe that

$$\mathcal{F}_{s,\lambda}\left(\mathit{Close}_{\frac{\lambda}{s}}(S)\right) - \mathcal{F}_{s,\lambda}(S) = P\left(\mathit{Close}_{\frac{\lambda}{s}}(S)\right) + \frac{s}{\lambda}\left|\mathit{Close}_{\frac{\lambda}{s}}(S) \setminus S\right| - P(S).$$

Setting $s = s_2(\lambda)$ and $r = \frac{\lambda}{s_2(\lambda)}$ in the above equality and observing that $\frac{\lambda}{s_2(\lambda)}$ is an increasing function of λ we have that

$$\mathcal{F}_{s_2(\lambda),\lambda}\left(\mathit{Close}_{\frac{\lambda}{s_2(\lambda)}}(S)\right) - \mathcal{F}_{s_2(\lambda),\lambda}(S)$$

is > 0 (resp. $= 0$, < 0) if and only if $\lambda < \lambda_1$ (resp. $\lambda = \lambda_1$, $\lambda > \lambda_1$).

Notice that $\lambda_1 = 0$ if and only if $R_1 = 0$ and we have proved that this happens if and only if $R_c(S) = 0$.

(b) (i) We are assuming that $P(co(S)) < \frac{P(S_1)}{|S_1|}|S|$. Let us first assume that the radius of S_1 is $>$ than the radius of S_2, hence $\frac{P(S)}{|S|} > \frac{P(S_1)}{|S_1|}$. If $R_c(S) > 0$, then

$$\begin{aligned} P\left(\mathit{Close}_r(S)\right) + \tfrac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right| &\to P\left(\mathit{Close}_{R_c(S)}(S)\right) + \tfrac{1}{R_c(S)}\left|\mathit{Close}_{R_c(S)}(S) \setminus S\right| \\ &> P(S) > \tfrac{P(S_1)}{|S_1|}|S| \qquad \text{as } r \to R_c(S) \end{aligned}$$

and

$$P\left(\mathit{Close}_r(S)\right) + \frac{1}{r}\left|\mathit{Close}_r(S) \setminus S\right| \to P(S) > \frac{P(S_1)}{|S_1|}|S| \qquad \text{as } r \to 0+$$

in case that $R_c(S) = 0$. On the other hand

$$P\left(Close_r\left(S\right)\right) + \frac{1}{r}\left|Close_r\left(S\right) \setminus S\right| \to P(co(S)) < \frac{P(S_1)}{|S_1|}|S|. \qquad \text{as } r \to \infty\ .$$

Thus there is a unique value $R_2 \in (R_c(S), \infty)$ satisfying (16).
Now, if $\frac{P(S)}{|S|} = \frac{P(S_1)}{|S_1|}$, then both equations (14) and (16) are the same and we can take $R_2 = R_1$. Hence $\lambda_2 = \lambda_1$. Clearly, if $R_2 = R_1$, then $\frac{P(S)}{|S|} = \frac{P(S_1)}{|S_1|}$. Notice that if $\lambda_1 = \lambda_2$, then

$$0 \geq (R_1 - R_2)|S_1||S_2| = R_1 R_2(|S_1|P(S_2) - |S_2|P(S_1)) \geq 0.$$

Thus $R_1 = R_2$. Note that if $\frac{P(S)}{|S|} = \frac{P(S_1)}{|S_1|}$ and $R_c(S) = 0$, by (i), $R_2 = R_1 = 0$ and $\lambda_2 = \lambda_1 = 0$.
To prove (17) we proceed as in the proof of (i). The fact that $\lambda_2 \geq \lambda_1$ follows since $\frac{P(S_1)}{|S_1|} \leq \frac{P(S)}{|S|}$. From the explicit formula for λ_2, it follows that $\lambda_2 \leq \frac{P(S_1)}{|S_1|}$.

(ii) For $\lambda \in \left(\frac{|S_2|}{P(S_2)}, \frac{|S_1|}{P(S_1)}\right)$ the only possible minimizers for $\mathcal{F}_{0,\lambda}$ are S_1 and $\Gamma_{0,\lambda}$. For $\lambda = \frac{|S_2|}{P(S_2)}$, $\Gamma_{0,\lambda}$ is a minimizer.
Proposition 4.1 states that for $\lambda > \frac{|S_1|}{P(S_1)}$, $\emptyset$ is the only possible minimizer. Since $\lambda \to \min_X \mathcal{F}_{0,\lambda}(X)$ is continuous, there has to be $\lambda = \lambda_2 \in \left(\frac{|S_2|}{P(S_2)}, \frac{|S_1|}{P(S_1)}\right)$ such that $\mathcal{F}_{0,\lambda_2}(S_1) = \mathcal{F}_{0,\lambda_2}\left(\Gamma_{0,\lambda_2}(S)\right)$. Rearrangement of this equation gives

$$\lambda_2 = \frac{\left|\Gamma_{0,\lambda_2}(S) \cap S_2\right|}{P\left(\Gamma_{0,\lambda_2}(S)\right) - P\left(S_1\right)}\ .$$

□

6 Proof of Theorem 5.8

We consider the three different intervals of λ. For each of them we compute $C_{s,\lambda}$ for $s \in [0, 1]$.

(a) $\lambda \in [0, \lambda_1]$. In this case $\lambda < r_2$ such that $\Gamma_{s,\lambda}(S) = Close_{\frac{\lambda}{s}}(S)$.

(a1) Let us prove that there is a function $s_a(\lambda)$, $\lambda \in [0, \lambda_1]$, such that

$$\mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) < \mathcal{F}_{s,\lambda}(S) \qquad \text{(respectively $=$, $>$)} \tag{23}$$

if and only if $s \in [0, s_a(\lambda))$, resp. $s = s_a(\lambda)$, $s > s_a(\lambda)$. Notice that $\mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) \leq \mathcal{F}_{s,\lambda}(S)$ if and only if

$$P\left(Close_{\frac{\lambda}{s}}(S)\right) + \frac{s}{\lambda}\left|Close_{\frac{\lambda}{s}}(S) \setminus S\right| \leq P(S)\ . \tag{24}$$

Clearly, by Proposition 5.6 (a), if we define

$$s_a(\lambda) = \frac{1}{R_1}\lambda \quad \lambda \in [0, \lambda_1] \ ,$$

then the equality in (24) holds identically. Now, by Lemma 5.11 (iii), the left hand side of (24) is an increasing function of s, and the identity in (24) only holds at $s = s_a(\lambda)$. Thus (23) holds.

Remark that (23) holds for any value of λ.

(a2) *Identification of* $C_{s,\lambda}$. Recall that, by Lemma 5.5, for any $s \in \left[0, 1 - \lambda\frac{P(S_2)}{|S_2|}\right]$ we have

$$\min\left\{\mathcal{F}_{s,\lambda}(S), \mathcal{F}_{s,\lambda}(S_1), \mathcal{F}_{s,\lambda}(S_2), 0\right\} = \mathcal{F}_{s,\lambda}(S) .$$

Thus $C_{s,\lambda} = Close_{\frac{\lambda}{s}}(S)$ if $s \in [0, s_a(\lambda)]$, $C_{s,\lambda} = S$ if $s \in \left(s_a(\lambda), 1 - \lambda\frac{P(S_2)}{|S_2|}\right]$. Notice that if $s = s_a(\lambda)$, S is also a minimizer of $\mathcal{F}_{s,\lambda}$.

Using (23) and Lemma 5.5 we clearly have that $C_{s,\lambda} = S_1$ if $s \in \left(1 - \lambda\frac{P(S_2)}{|S_2|}, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$ and $C_{s,\lambda} = \emptyset$ if $s > 1 - \lambda\frac{P(S_1)}{|S_1|}$.

(b) Let $\lambda \in (\lambda_1, \lambda_2]$. In this case, let us prove that there is a function $s_b(\lambda)$ such that

$$\mathcal{F}_{s_b(\lambda),\lambda}\left(\Gamma_{s_b(\lambda),\lambda}(S)\right) = \mathcal{F}_{s_b(\lambda),\lambda}(S_1) \qquad \lambda \in [\lambda_1, \lambda_2] \ . \tag{25}$$

Let us consider two cases $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$ and $\frac{|S_1|}{P(S_1)} \geq \frac{|S|}{P(\mathrm{co}(S))}$.

(b1) In this case we assume that $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$.

i) *Proof of* (25). Recall from Lemma 5.12 that in this situation we have $\Gamma_{s,\lambda}(S) = Close_{\frac{\lambda}{s}}(S)$. In this case $\lambda_2 = \frac{R_2|S_1|}{R_2P(S_1)+|S_1|}$. We have

$$s_a(\lambda) = \frac{\lambda}{R_1} \leq 1 - \lambda\frac{P(S_1)}{|S_1|}.$$

if and only if

$$\lambda \leq \frac{R_1|S_1|}{R_1P(S_1) + |S_1|} := \bar{\lambda}_1.$$

Observe that $\bar{\lambda}_1 \leq \lambda_2$ since $R_1 \leq R_2$.

Let us work in the interval $s \in \left[0, \frac{\lambda}{R_1}\right]$ for all $\lambda \in [\lambda_1, \lambda_2]$. Let us prove that

$$\mathcal{F}_{0,\lambda}(\mathrm{co}(S)) < \mathcal{F}_{0,\lambda}(S_1) \ . \tag{26}$$

Indeed, since $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$, after some simple computations we deduce that

$$\lambda_2 = \frac{R_2|S_1|}{R_2P(S_1) + |S_1|} < \frac{|S_2|}{P(\mathrm{co}(S)) - P(S_1)}.$$

Thus, if $\lambda \leq \lambda_2$, then $\lambda < \frac{|S_2|}{P(\mathrm{co}(S))-P(S_1)}$ and this is equivalent to (26).
Moreover, by Proposition 5.6 (b) (assuming that $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$), for $\lambda < \lambda_2$ and $s = s_1(\lambda) = 1 - \lambda\frac{P(S_1)}{|S_1|}$ we have

$$\mathcal{F}_{s_1(\lambda),\lambda}(S_1) < \mathcal{F}_{s_1(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_1(\lambda)}}(S)\right),$$

with equality if $\lambda = \lambda_2$, and for $\lambda \in (\lambda_1, \lambda_2]$ and $s = \frac{\lambda}{R_1}$, we have

$$\mathcal{F}_{\frac{\lambda}{R_1},\lambda}(S_1) < \mathcal{F}_{\frac{\lambda}{R_1},\lambda}(S) = \mathcal{F}_{\frac{\lambda}{R_1},\lambda}(Close_{R_1}(S)) \tag{27}$$

(the first inequality being true because $\lambda > \lambda_1$). Since both functions $s \to \mathcal{F}_{s,\lambda}(S_1)$ and $s \to \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right)$ are continuous in s, they have to intersect for some $s \in \left[0, \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}\right]$. Hence there is at least one value s that satisfies (25). Let $s_b(\lambda)$ be the smallest value of s satisfying (25).
Notice that we have that $s_b(\lambda) < \frac{\lambda}{R_1}$ for any $\lambda \in (\lambda_1, \lambda_2]$ and $s_b(\lambda) < 1 - \lambda\frac{P(S_1)}{|S_1|}$ for any $\lambda < \lambda_2$ (with equality if $\lambda = \lambda_2$).

ii) We show that $s_b(\lambda)$ is the unique value of $s \in \left[0, \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}\right]$ satisfying (25), if $\lambda \in (\lambda_1, \lambda_2]$.
Clearly, if $\lambda = \lambda_2$, then $s_b(\lambda) = 1 - \lambda\frac{P(S_1)}{|S_1|} = \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}$ and (25) holds. Our assertion is true in this case.
Assume that $\lambda \in (\lambda_1, \lambda_2)$. Let us prove that for any $s \in \left(s_b(\lambda), \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}\right]$ we have

$$\mathcal{F}_{s,\lambda}(S_1) < \mathcal{F}_{s,\lambda}(\mathrm{Close}_{\frac{\lambda}{s}}(S)) . \tag{28}$$

Suppose that we find $s_b(\lambda) < t_1 < t_2 < \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}$ where

$$\mathcal{F}_{t_1,\lambda}(S_1) < \mathcal{F}_{t_1,\lambda}(\mathrm{Close}_{\frac{\lambda}{t_1}}(S)) , \tag{29}$$

$$\mathcal{F}_{t_2,\lambda}(S_1) > \mathcal{F}_{t_2,\lambda}(\mathrm{Close}_{\frac{\lambda}{t_2}}(S)) . \tag{30}$$

Let us compute $C_{t_1,\lambda}$. Observe that by (29) S_1 has less energy than $\Gamma_{t_1,\lambda}(S)$, and $\Gamma_{t_1,\lambda}(S)$ is better than S because $t_1 < \frac{\lambda}{R_1}$. Also $\mathcal{F}_{t_1,\lambda}(S_1) < 0$ because $t_1 < 1 - \lambda\frac{P(S_1)}{|S_1|}$. Thus $\mathcal{F}_{t_1,\lambda}(S) = \mathcal{F}_{t_1,\lambda}(S_1) + \mathcal{F}_{t_1,\lambda}(S_2) \leq \mathcal{F}_{t_1,\lambda}(S_2)$. Thus $C_{t_1,\lambda} = S_1$.
Let us compute $C_{t_2,\lambda}$. Observe that $\Gamma_{t_2,\lambda}(S)$ is better than S_1 (by (30)). And $\Gamma_{t_2,\lambda}(S)$ is better than S because $t_2 < \frac{\lambda}{R_1}$. Also $\mathcal{F}_{t_2,\lambda}(S_1) \leq 0$ because $t_2 < 1 - \lambda\frac{P(S_1)}{|S_1|}$. Thus $\mathcal{F}_{t_2,\lambda}(S) = \mathcal{F}_{t_2,\lambda}(S_1) + \mathcal{F}_{t_2,\lambda}(S_2) \leq \mathcal{F}_{t_2,\lambda}(S_2)$. Thus $C_{t_2,\lambda} = \Gamma_{t_2,\lambda}(S)$.
It is not possible that $t_1 < t_2$ and the optimal set $C_{t_2,\lambda}$ contains the optimal set $C_{t_1,\lambda}$. We conclude that $\mathcal{F}_{s,\lambda}(S_1) \leq \mathcal{F}_{s,\lambda}(\Gamma_{s,\lambda}(S))$ for all $s \in \left(s_b(\lambda), \min\left\{\frac{\lambda}{R_1}, 1-\lambda\frac{P(S_1)}{|S_1|}\right\}\right)$.

The inequality has to be strict. Otherwise there would be two points $t < t'$ such that the minima of $\mathcal{F}_{t,\lambda}$ are both S_1 and $\Gamma_{t,\lambda}(S)$ and the minima of $\mathcal{F}_{t',\lambda}$ are both S_1 and $\Gamma_{t',\lambda}(S)$. Then S_1 would contain $\Gamma_{t',\lambda}(S)$, a contradiction. Thus (28) is proved.

iii) *Computation of* $C_{s,\lambda}$ *for* $\lambda \in (\lambda_1, \lambda_2]$ *and any* s.

If $s \in [0, s_b(\lambda))$ we argue as for t_2 and we deduce that the optimum is $\mathrm{Close}_{\frac{\lambda}{s}}(S)$. Letting $s \to s_b(\lambda)$ we deduce that $C_{s_b(\lambda),\lambda} = \mathrm{Close}_{\frac{\lambda}{s_b(\lambda)}}(S)$.

If $s \in \left(s_b(\lambda), \min\left\{\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}\right\}\right]$ we argue as for t_1 and we deduce that the optimum is S_1.

If $\min\left\{\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}\right\} = \frac{\lambda}{R_1}$, i.e., if $\lambda \in (\lambda_1, \bar{\lambda}_1]$, let us compute the optimum for $s \in (\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}]$. On this interval $\mathcal{F}_{s,\lambda}(S_1) \leq 0$. Thus $\mathcal{F}_{s,\lambda}(S) = \mathcal{F}_{s,\lambda}(S_1) + \mathcal{F}_{s,\lambda}(S_2) \leq \mathcal{F}_{s,\lambda}(S_2)$. Also (by the definition of R_1) on this interval $\mathcal{F}_{s,\lambda}(\mathrm{Close}_{\frac{\lambda}{s}}(S)) > \mathcal{F}_{s,\lambda}(S)$. Thus the optimum is either S_1 or S. By monotonicity of the optimum with respect to s and the fact that the optimum in $(s_b(\lambda), \frac{\lambda}{R_1}]$ is S_1, we deduce that it is also S_1 in $(\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}]$.

If $\min\left\{\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}\right\} = \frac{\lambda}{R_1}$ and $s > 1 - \lambda\frac{P(S_1)}{|S_1|}$, we have that $\mathcal{F}_{s,\lambda}(S_1) > 0$ and the minimum is $\emptyset$.

If $\min\left\{\frac{\lambda}{R_1}, 1 - \lambda\frac{P(S_1)}{|S_1|}\right\} = 1 - \lambda\frac{P(S_1)}{|S_1|}$, i.e., if $\lambda \in (\bar{\lambda}_1, \lambda_2]$, as in the previous paragraph the minimum for $s > 1 - \lambda\frac{P(S_1)}{|S_1|}$ is $\emptyset$.

Let us point out that for $\lambda = \lambda_2$ and for $s > s_b(\lambda_2) = 1 - \lambda_2\frac{P(S_1)}{|S_1|}$, we have that

$$\mathcal{F}_{s,\lambda_2}(\mathrm{Close}_{\frac{\lambda_2}{s}}(S)) > \mathcal{F}_{s_b(\lambda_2),\lambda_2}(\mathrm{Close}_{\frac{\lambda_2}{s_b(\lambda_2)}}(S)) = \mathcal{F}_{s_b(\lambda_2),\lambda_2}(S_1) = 0.$$

Since also $\mathcal{F}_{s,\lambda_2}(S_1) > 0$, then $C_{s,\lambda_2} = \emptyset$.

(b2) Assume that $\frac{|S_1|}{P(S_1)} \geq \frac{|S|}{P(\mathrm{co}(S))}$.

i) *Define* $s_b(\lambda)$ *for* $\lambda \in (\lambda_1, \lambda_2]$. In this case $\lambda_2 = \frac{|S_2|}{P(\mathrm{co}(S)) - P(S_1)}$.

Since $\lambda \leq \lambda_2 = \frac{|S_2|}{P(\mathrm{co}(S) - P(S_1)}$, then (26) holds. On the other hand (27) also holds for any $\lambda \in (\lambda_1, \lambda_2]$. As in paragraph (b1) we identify a solution $s_b(\lambda) \in \left[0, \frac{\lambda}{R_1}\right]$ of (25). Let us take the smallest one. Let us prove that $s_b(\lambda) \in \left[0, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$. Let $s = s_1(\lambda) = 1 - \lambda\frac{P(S_1)}{|S_1|}$. Then $\mathcal{F}_{s_1(\lambda),\lambda}(S_1) = 0$ and

$$\mathcal{F}_{s_1(\lambda),\lambda}(\mathrm{Close}_{\frac{\lambda}{s_1(\lambda)}}(S)) = P(\mathrm{Close}_{\frac{\lambda}{s_1(\lambda)}}(S)) + \frac{s_1(\lambda)}{\lambda}|\mathrm{Close}_{\frac{\lambda}{s_1(\lambda)}}(S) \setminus S| - \frac{P(S_1)}{|S_1|}|S|$$

$$\geq P(\mathrm{co}(S)) - \frac{P(S_1)}{|S_1|}|S| \;\geq\; 0\,.$$

Notice that the second inequality is strict if $\frac{|S_1|}{P(S_1)} > \frac{|S|}{P(\mathrm{co}(S))}$, while the first is strict if $\frac{|S_1|}{P(S_1)} = \frac{|S|}{P(\mathrm{co}(S))}$ and $s_1(\lambda) > 0$. In both cases, since $s_b(\lambda)$ is the smallest solution of (25), we have $s_b(\lambda) \in [0, 1 - \lambda\frac{P(S_1)}{|S_1|}]$.
If $\frac{|S_1|}{P(S_1)} = \frac{|S|}{P(\mathrm{co}(S))}$, and $s_1(\lambda) = 0$, then $\lambda = \frac{|S_1|}{P(S_1)} = \lambda_2$ and

$$\mathcal{F}_{s_1(\lambda_2),\lambda_2}\left(\Gamma_{s_1(\lambda_2),\lambda_2}(S)\right) = \mathcal{F}_{s_1(\lambda_2),\lambda_2}(S_1) = 0,$$

and we take $s_b(\lambda_2) = 0$.

ii) *Computation of $C_{s,\lambda}$ for $\lambda \in (\lambda_1, \lambda_2]$ and any s.*
If $\lambda \in (\lambda_1, \bar{\lambda}_1]$, then $s_b(\lambda) \le \frac{\lambda}{R_1} \le 1 - \lambda\frac{P(S_1)}{|S_1|}$. Then the argument is identical to the same case in Step iii).
If $\lambda \in (\bar{\lambda}_1, \lambda_2]$, then $1 - \lambda\frac{P(S_1)}{|S_1|} < \frac{\lambda}{R_1}$. We have that $\mathcal{F}_{s,\lambda}(\Gamma_{s,\lambda}(S)) \le \mathcal{F}_{s,\lambda}(S)$ for all $s \in \left[0, \frac{\lambda}{R_1}\right]$. If $s \in \left[0, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$ we have $\mathcal{F}_{s,\lambda}(S_1) \le 0$. Then $\mathcal{F}_{s,\lambda}(S) = \mathcal{F}_{s,\lambda}(S_1) + \mathcal{F}_{s,\lambda}(S_2) \le \mathcal{F}_{s,\lambda}(S_2)$. Thus the optimal set for $s \in \left[0, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$ can be only $\Gamma_{s,\lambda}(S)$ or S_1.
As in step ii) we prove that $s_b(\lambda)$ is the unique value of $s \in \left[0, \frac{\lambda}{R_1}\right]$ satisfying (25) if $\lambda \in (\lambda_1, \lambda_2]$. Then we proceed as in Step iii) to prove that $C_{s,\lambda} = \Gamma_{s,\lambda}(S)$ if $s \in [0, s_b(\lambda)]$, and $C_{s,\lambda} = S_1$ if $s \in \left(s_b(\lambda), 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$. For $s > 1 - \lambda\frac{P(S_1)}{|S_1|}$, $C_{s,\lambda} = \emptyset$ since $\mathcal{F}_{s,\lambda}(S_1) > 0$.

(c) Let $\lambda \in (\lambda_2, \lambda_3]$. Again we distinguish two cases $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$ and $\frac{|S_1|}{P(S_1)} \ge \frac{|S|}{P(\mathrm{co}(S))}$.

(c1) *Assume that $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$.*

i) Recall from Lemma 5.12 we have again that $\Gamma_{s,\lambda}(S) = \mathit{Close}_{\frac{\lambda}{s}}(S)$. In this case $\lambda_2 = \frac{R_2|S_1|}{R_2 P(S_1) + |S_1|}$ and $\lambda_3 = \frac{|S|}{P(\mathrm{co}(S))}$. We look for a value $s_c(\lambda)$, $\lambda \in (\lambda_2, \lambda_3]$, such that

$$\mathcal{F}_{s_c(\lambda),\lambda}\left(\mathit{Close}_{\frac{\lambda}{s_c(\lambda)}}(S)\right) = 0. \tag{31}$$

Let $\lambda \in (\lambda_2, \lambda_3)$. If $s = 0$, we have

$$\mathcal{F}_{0,\lambda}(co(S)) < 0.$$

Let $s_3(\lambda) = 1 - \lambda\frac{P(co(S))}{|S|}$. Since by Lemma 5.11 $\lambda \to \mathcal{F}_{s_3(\lambda),\lambda}\left(\mathit{Close}_{\frac{\lambda}{s_3(\lambda)}}(S)\right)$ is strictly decreasing, then

$$0 = \mathcal{F}_{0,\frac{|S|}{P(co(S))}}(co(S)) < \mathcal{F}_{s_3(\lambda),\lambda}\left(\mathit{Close}_{\frac{\lambda}{s_3(\lambda)}}(S)\right).$$

Then for any $\lambda \in (\lambda_2, \lambda_3)$ there exists $s_c(\lambda) < s_3(\lambda)$ such that (31) holds.

Let $\lambda = \lambda_3$. Then $\mathcal{F}_{0,\lambda_3}(co(S)) = 0$ and $s_3(\lambda_3) = 0$. Hence $\mathcal{F}_{s_3(\lambda),\lambda}\left(Close_{\frac{\lambda}{s_3(\lambda)}}(S)\right) = 0$. Since $\mathcal{F}_{s,\lambda_3}\left(Close_{\frac{\lambda_3}{s}}(S)\right)$ is strictly increasing in s we have that $\mathcal{F}_{s,\lambda_3}\left(Close_{\frac{\lambda_3}{s}}(S)\right) > 0$ for any $s \in \left(0, \frac{\lambda_3}{R_c(S)}\right]$. Then $s_c(\lambda_3) = 0 = s_3(\lambda_3)$.

ii) Let us prove that

$$\text{if } \lambda \geq \lambda_2 = \tfrac{R_2|S_1|}{R_2P(S_1)+|S_1|} \text{ and } s \in \left[0, 1-\lambda\tfrac{P(S_1)}{|S_1|}\right], \text{ then } \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) \leq \mathcal{F}_{s,\lambda}(S_1). \tag{32}$$

Indeed, if $\lambda \geq \lambda_2$ and $s \in \left[0, 1-\lambda\frac{P(S_1)}{|S_1|}\right]$, then $s \leq 1-\lambda\frac{P(S_1)}{|S_1|} \leq \frac{\lambda}{R_2}$. That is, $\frac{s}{\lambda} \leq \frac{1}{R_2}$. Then, by Lemma 5.11 (iii), we have

$$P(\mathrm{Close}_{\frac{\lambda}{s}}(S)) + \frac{s}{\lambda}|\mathrm{Close}_{\frac{\lambda}{s}}(S)\backslash S| \leq P(\mathrm{Close}_{R_2}(S)) + \frac{1}{R_2}|\mathrm{Close}_{R_2}(S)\backslash S| = \frac{P(S_1)}{|S_1|}|S|. \tag{33}$$

Now, $\mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) \leq \mathcal{F}_{s,\lambda}(S_1)$ if and only if

$$P(\mathrm{Close}_{\frac{\lambda}{s}}(S)) + \frac{s}{\lambda}|\mathrm{Close}_{\frac{\lambda}{s}}(S)\backslash S| \leq P(S_1) - \frac{1-s}{\lambda}|S_1| + \frac{1-s}{\lambda}|S| = P(S_1) + \frac{1-s}{\lambda}|S_2|.$$

Thus, by (33), it is sufficient to prove that

$$\frac{P(S_1)}{|S_1|}|S| \leq P(S_1) + \frac{1-s}{\lambda}|S_2|.$$

But this is true since $\frac{P(S_1)}{|S_1|} \leq \frac{1-s}{\lambda}$. Hence (32) holds.
Since $\mathcal{F}_{s,\lambda}(S_1) < 0$ for $s \in \left[0, 1-\lambda\frac{P(S_1)}{|S_1|}\right)$, then (32) implies

$$\text{if } \lambda \geq \lambda_2 = \tfrac{R_2|S_1|}{R_2P(S_1)+|S_1|} \text{ and } s \in \left[0, 1-\lambda\tfrac{P(S_1)}{|S_1|}\right), \text{ then } \mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) < 0.$$

In particular, we have that $s_c(\lambda) > 1-\lambda\frac{P(S_1)}{|S_1|}$.
Observe also that the inequality in (32) is strict if $s \in \left[0, 1-\lambda\frac{P(S_1)}{|S_1|}\right)$.

iii) Let us prove that the optimum in $[0, 1-\lambda\frac{P(S_1)}{|S_1|}]$ is $\mathrm{Close}_{\frac{\lambda}{s}}(S)$. By (32) it cannot be S_1. On the other hand, by the first paragraph of Step b1, if $\lambda \in (\lambda_2, \lambda_3]$, we have that $\lambda > \lambda_2 \geq \bar{\lambda}_1$ and, therefore, $\frac{\lambda}{R_1} > 1-\lambda\frac{P(S_1)}{|S_1|}$. By the last remark in Step (a1), $\mathcal{F}_{s,\lambda}\left(Close_{\frac{\lambda}{s}}(S)\right) \leq \mathcal{F}_{s,\lambda}(S)$ in the interval $[0, 1-\lambda\frac{P(S_1)}{|S_1|}]$. On the other hand, on that interval, $\mathcal{F}_{s,\lambda}(S) \leq \mathcal{F}_{s,\lambda}(S_2)$. Thus, the optimum is $\mathrm{Close}_{\frac{\lambda}{s}}(S)$ (its energy being negative).
Again, the optimum in $(1-\lambda\frac{P(S_1)}{|S_1|}, s_c(\lambda)]$ is $\mathrm{Close}_{\frac{\lambda}{s}}(S)$. The optimum cannot be S_1 because its energy is positive. By the assumption $r_2 \leq r_1$, the energy of S_2 is also positive. Thus, also is for S. The optimum is $\mathrm{Close}_{\frac{\lambda}{s}}(S)$.

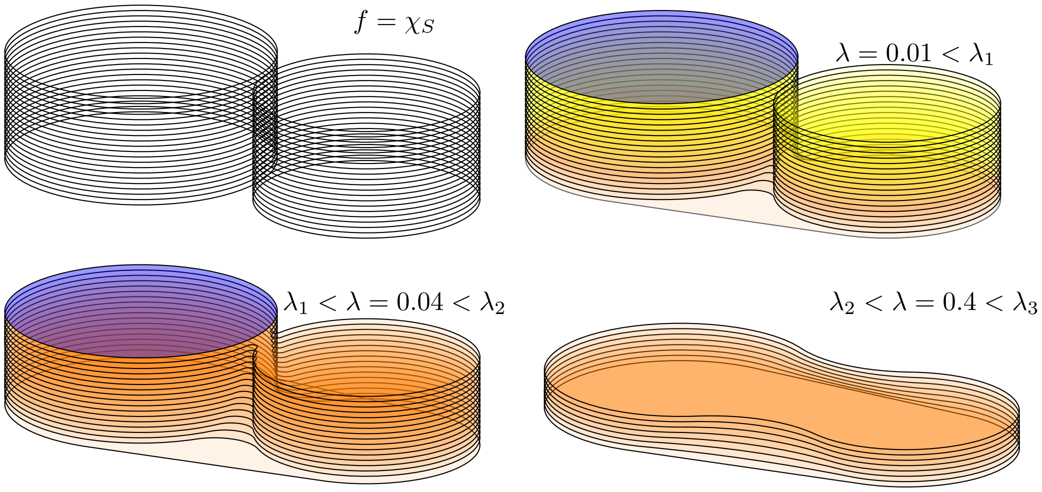

Figure 11: Levelsets of the solutions u_λ for different values of λ. S_1, S_2 are balls with radius $r_1 = 1.2, r_2 = 1$, the distance between them is $d = 0.05$. In this case $\frac{|S_1|}{P(S_1)} < \frac{|S|}{P(\mathrm{co}(S))}$.

Notice that the argument in the previous paragraph shows that for $s > s_c(\lambda)$ the optimum can only be either $\Gamma_{s,\lambda}(S)$ or $\emptyset$. Thus, either there is a maximal interval of values of s, say $(s_c(\lambda), s_d(\lambda)]$, not reduced to $s_c(\lambda)$, where (31) holds, in which case $\mathrm{Close}_{\frac{\lambda}{s}}(S)$ is the optimum up to $s_d(\lambda)$, or the energy of $\mathrm{Close}_{\frac{\lambda}{s}}(S)$ becomes positive immediately after $s_c(\lambda)$ and the optimum is $\emptyset$. Thus, we may take $s_c(\lambda)$ as the maximal solution of (31).

(c2) *Assume that* $\frac{|S|}{P(\mathrm{co}(S))} \leq \frac{|S_1|}{P(S_1)}$.

In this case, $\lambda_2 = \frac{|S_2|}{P(\mathrm{co}(S)) - P(S_1)}$. Since $P(\mathrm{co}(S)) < P(S)$, we have $\mathcal{F}_{0,\lambda}(\mathrm{co}(S)) < \mathcal{F}_{0,\lambda}(S)$. If we take $\lambda > \lambda_2$, we have $\mathcal{F}_{0,\lambda}(S_1) < \mathcal{F}_{0,\lambda}(\mathrm{co}(S))$. Thus, for $s \in \left[0, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$ small enough the optimum is S_1. By monotonicity of the level sets of u_λ, S_1 is the optimum for all $s \in \left[0, 1 - \lambda\frac{P(S_1)}{|S_1|}\right]$. For $s > 1 - \lambda\frac{P(S_1)}{|S_1|}$ is the emptyset.

(d) Since $\lambda > \lambda_3 = \|\chi_S\|_*$, $C_{s,\lambda} = \emptyset$ by Proposition 4.1. This concludes the proof.

□

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