A survey on the existence of isoperimetric sets in the space $\mathbb{R}^N$ with density

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July 15, 2014

Abstract

The aim of this survey is to give a precise idea of the recent results on existence of isoperimetric sets in $\mathbb{R}^N$ with density. We will mainly focus on the overall ideas, leaving away some technical details of the proofs, which can be found in the cited papers. No previous knowledge on the subject is assumed from the reader.

This survey originates from a talk of the author at the conference "New Trends in Nonlinear PDE's" held at the Accademia dei Lincei on November 26th, 2013. I wish to dedicate this paper to Carlo Sbordone, because of his recent 65th birthday, and to Ula, because she will become my wife in few days.

1 Introduction

The aim of this short survey is to discuss the more recent results on the existence of isoperimetric sets in the space $\mathbb{R}^N$ with density. The problem is very easy to state: for a given $L^1_{\rm loc}$ and l.s.c. function $f:\mathbb{R}^N\to\mathbb{R}^+$, one defines the *generalized volume* and the *generalized perimeter* of a Borel set $\Omega\subseteq\mathbb{R}^N$ as

$$|\Omega| = \int_\Omega f(x)\,dx\,, \qquad\qquad P(\Omega) = \int_{\partial^*\Omega} f(y)\,d\mathscr{H}^{N-1}(y)\,,$$

where the reduced boundary $\partial^*\Omega$ of Ω coincides with the usual topological boundary if the set Ω is smooth enough. To read this survey there is no need to know

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what exactly the reduced boundary is, however the interested reader can find all the definitions and main properties for instance in [1].

The *isoperimetric problem* consists then, as always, in trying to minimize the perimeter of sets with a fixed volume. Of course, this coincides with the classical (or "Euclidean") isoperimetric problem when $f \equiv 1$, but otherwise a number of different possibilities arise. This problem is estremely well-known and deeply studied for a number of reasons; the interested reader can find some history, explanations and a large bibliography for instance in the papers [5, 3]. Let us now start by discussing, in a very informal way, what should happen about this problem, and which are the most interesting questions.

First of all, it is very simple to understand that, in general, one should not expect existence of isoperimetric sets (i.e., sets minimizing the perimeter for their volume). Indeed, starting with any positive density f, and with any sequence of sets with constant volume, it is possible to lower the value of f on the boundaries of these sets: this will not affect the volumes of the sets, but then the perimeters can be made arbitrarily small. Notice that, in this way, the function f remains lower semi-continuous (actually, if one wants only to consider continuous densities, then the same argument applies with minor modifications).

As a consequence, it is clear that the first important task is to obtain conditions under which the existence of isoperimetric sets is ensured, and the goal of this survey is to discuss precisely this question. Of course, other interesting questions concern the regularity and other geometrical properties of the isoperimetric sets. In the remaining of the introduction, we will explain some simple but fundamental properties of the problem, and then we will give the claim of the main result that we are going to present.

1.1 Preliminaries on the problem: the mass which escapes at infinity

Let us immediately recall a very basic lower semi-continuity result, which directly comes from the analogous result for BV functions (see for instance [1]).

Lemma 1.1. *Let $\{\Omega_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^N$ be a sequence of sets such that the characteristic functions χ_{Ω_j} weakly converge in the $BV_{\rm loc}$ sense to χ_Ω for some $\Omega \subseteq \mathbb{R}^N$. Then, $P(\Omega) \leq \liminf P(\Omega_j)$.*

Since standard compactness results ensure that, from any sequence $\{\Omega_j\}$ of sets, it is possible to extract a subsequence such that the sets χ_{Ω_j} converge to some characteristic function χ_Ω, the above lemma could seem to give immediately a general existence result: indeed, for any given volume V one can always find a sequence which tends to minimize the perimeter (such a sequence is called "isoperimetric sequence"), and extract a converging subsequence, hoping that the limit will be an isoperimetric set. Unfortunately, this does not work so easily, becuase the limiting set Ω could happen to have a strictly smaller volume, hence it would be not a competitor for the isoperimetric problem with volume V. Actually, for any given bounded domain $D \subseteq \mathbb{R}^N$ one has that the volume of Ω in D coincides with the limit of the volumes of Ω_j in D; in other words, the only risk is that the sequence Ω_j "loses some mass at infinity". For instance, if the sets Ω_j are all balls of unit volume whose centers go to infinity, then the limit is the empty set, which clearly does not have unit volume. As a consequence of the above observations, the following results are straightforward to prove.

Theorem 1.2. *If the volume of the whole $\mathbb{R}^N$ is finite, say $|\mathbb{R}^N| = M$, then there exists an isoperimetric set for every volume $V \leq M$. If the volume of $\mathbb{R}^N$ is infinite, but for some volume $V > 0$ there exists a bounded isoperimetric sequence relative to volume V, then an isoperimetric set for volume V exists.*

The idea of proof is extremely simple: if $\mathbb{R}^N$ has only a finite volume, then it is not possible that some mass escapes at infinity, because "there is no space at infinity". More precisely, for any positive $\varepsilon > 0$ one can find a big domain D such that, out of D, there is only a volume smaller than ε. As a consequence, taking an isoperimetric sequence Ω_j and calling Ω a limiting set of (a suitable subsequence of) $\{\Omega_j\}$, then the volume of Ω is bigger than $V - \varepsilon$ for any positive ε, and then $|\Omega| = V$ and so Ω is isoperimetric, as said above. In $\mathbb{R}^N$ has infinite volume, but there is an isoperimetric sequence completely contained in some big bounded domain D, then again the volume of Ω is exactly V, because the mass is not escaping at infinity, being confined in D.

These simple facts already allow us to do some interesting observations. A first one concerns the cases when the volume of $\mathbb{R}^N$ is finite: in all these cases, the study of existence is useless because existence is always automatically true. This does not mean that the isoperimetric problem is not interesting (for instance, the Gaussian density $f(x) = e^{-|x|^2/2}$ is extremely studied), but only that the interesting questions are not the existence. A second, deeper, one concerns the general case when $\mathbb{R}^N$ has infinite volume: to show the existence of isoperimetric sets of a given volume, one could try to show that isoperimetric sequences do not have any reason to escape at infinity. Even though this is a basic and obvious observation, a good strategy for showing the existence is precisely this one.

1.2 Preliminaries on the problem: the regions with high or low density

Let us now discuss what we should expect to happen in zones where the density f is high, or low. To start with, let us imagine that f is constantly C in a large region of $\mathbb{R}^N$, and seek for an isoperimetric set, say of volume 1, in that region. Since the density is constant, then of course the problem coincides with the usual Euclidean problem, up to a multiplicative constant, and then an isoperimetric set is simply a ball B_r of radius r and volume 1. Its volume and perimeter are then given by

$$1 = |B| = C\omega_N r^N\,, \qquad\qquad P(B) = CN\omega_N r^{N-1}\,,$$

which just by substituting gives

$$P(B) = C^{1/N} N \omega_N^{1/N}\,.$$

As a consequence, the perimeter of a ball of unit volume is higher when the constant C is higher. Notice that this is a consequence of the fact that volume scales with power N, and perimeter only with power $N-1$; in other words, to obtain a unit volume in a region where the constant C is small, one needs a much "bigger" ball (that is, a ball with a big radius), and of course a larger radius goes in the direction of a larger perimeter: however, the positive effect of the density being low is stronger than the negative effect of the radius being large. This is a simple but quite interesting information; suppose, indeed, that there are two big regions where the density is constant, and that the two constants are different: the above calculation suggests that, for the isoperimetric problem, placing a ball in the zone with low density is the better idea.

From this observation, one can get a general "rule", that is, *isoperimetric sets tend to privilege zones with lower density.* Of course, this is absolutely not a rule, and in fact it is in general false: the very best for a set would clearly be that the density is big inside the set (so the set can have a small dimension still reaching the desired volume) and small on the boundary of the set (so the set has a smaller perimeter); hence, if the density is rapidly changing then nothing can be easily said. However, keeping in mind the above "rule" is still useful, because in many cases this actually suggests the correct answer to the questions of existence or non existence, as we are going to see in a moment.

1.3 The main result of this survey

We are now going to present the main result of this survey, Theorem 1.4 below, which says that whenever a density is converging from below in the sense of Definition 1.3, isoperimetric sets exist for every volume. Before stating this result, we list some known existence of non-existence results, which are either trivial or can be found for instance in [5, 3]; basically, all these results should convince the reader that the case of densities converging from below treated here, is the only interesting one.

• A density which diverges at infinity: in this case, the "rule" would suggest that an isoperimetric sequence remains bounded, because there should be no gain for the isoperimetric problem in going where the density is big; since Theorem 1.2 ensures the existence of isoperimetric sets when isoperimetric sequences remain bounded, we can guess that existence should be true in this case. Actually, the existence is true only if the density is also radial: for a general diverging density it is still possible that existence fails.

• A density which goes to zero at infinity: in this case, the existence is automatically true if the volume of $\mathbb{R}^N$ is finite, thanks to Theorem 1.2; on the other hand, if the volume of $\mathbb{R}^N$ is infinite, the "rule" would suggests that any isoperimetric sequence goes to infinity, hence that no existence holds. And actually, it is true that existence always fails for densities which go to zero at infinity, of course under the assumption $|\mathbb{R}^N| = \infty$.

• A density for which $\liminf f(x) < \limsup f(x)$ at infinity: in this case, the density is "oscillating", hence one can expect that an isoperimetric sequence could either stay bounded or go to infinity, depending on how fast the density oscillates. And in fact, this intuition is correct: it is very easy to build examples of oscillating densities both with existence and with non-existence of isoperimetric sets, thus in this case one cannot find any general result.

• A density with a strictly positive and finite limit at infinity: this is the last possible case, and by the above discussion it is the only interesting one which is left, since in all the other cases either no result is true, or the true result is already known. Actually, it is immediately seen that this case should be divided in two subcases; indeed, if the density converges to the limit *from above*, then the "rule" would suggest that isoperimetric sequences might go to infinity, preventing the existence; on the other hand, if the density converges to its limit *from below*, then the suggestion would be that isoperimetric sequences are bounded, and then the existence should hold. To say this more formally, let us introduce the following notation.

Definition 1.3. *We say that the density $f : \mathbb{R}^N \to \mathbb{R}^+$ is* converging from below *to a limit $0 < \ell < +\infty$ if f converges to ℓ at infinity, and $f \leq \ell$ out of a sufficiently big ball.*

The above intuition can then be rephrased as follows: one could expect existence of isoperimetric sets for densities converging from below, and no general result for other densities (which are then either converging from above, or oscillating around the limit). The second guess is easily seen to be correct: it is simple to construct examples of densities which are converging to a limit $0 < \ell < \infty$, but not from below, both such that existence is true, and such that existence fails. Also the first guess is true, but the proof is quite more complicate, and the goal of this survey is precisely to describe it in good detail.

Theorem 1.4. *Let f be a density converging from below to $0 < \ell < \infty$. Then, for every volume $V > 0$, there exist isoperimetric sets of volume V.*

The proof of the above theorem is contained in the very recent paper [3]; the result was conjectured, and some particular cases were already proven, in the paper [5]. The plan of this survey is very simple: we collect some technical known facts and a basic definition in Section 2, and then Section 3 is devoted to describe in detail the proof of Theorem 1.4; our goal is not to give the completely formal proof, which is already contained in the above-cited paper, but to explain all the steps of the construction, proving almost formally most of them, in order to give a precise idea both to the initiated and to the non-initiated reader.

2 Some basic known facts and a definition

In this section we present a couple of known technical facts, and we give a useful definition. The first result shows that if an isoperimetric sequence weakly converges to a set, then this set is an isoperimetric set. Notice carefully that this seems in contrast with what we said right after Lemma 1.1, but it is not so: we are not saying that a weak limit Ω of an isoperimetric sequence corresponding to volume V is an isoperimetric set *for the volume* V, but that it is an isoperimetric set *for the volume* $|\Omega| \leq V$! In particular, we have noticed that an isoperimetric sequence could also vanish at infinity: in this case the lemma below just says that the empty set is an isoperimetric set for the problem with volume 0, which is of course emptily true.

Lemma 2.1. *Assume that $f \in L^1_{\rm loc}$ and that f is bounded from above and below far from the origin. Assume also that an isoperimetric sequence $\{\Omega_j\}$ of volume V weakly converges in L^1 to a set Ω. Then Ω is an isoperimetric set with volume $|\Omega|$.*

Proof. If $|\Omega| = 0$ there is nothing to prove, while if $|\Omega| = V$ then the claim is a direct consequence of Lemma 1.1; we can then assume without loss of generality that $0 < |\Omega| < V$.

Suppose now that Ω is not an isoperimetric set of volume $|\Omega|$: then, there exists F so that

$$|F| = |\Omega|\,, \qquad\qquad P(F) = P(\Omega) - \varepsilon\,,$$

for some $\varepsilon > 0$. By continuity, there exist a small constant $\delta > 0$ and a big radius $R > 0$ such that, for every $-\delta \leq t \leq \delta$, there exists a set F_t satisfying

$$F_t \subseteq B(0,R)\,, \qquad |F_t| = |\Omega| - t\,, \qquad P(F_t) \leq P(\Omega) - \frac{\varepsilon}{2}\,. \tag{2.1}$$

Up to increase R if necessary, we can also assume that

$$|\Omega \cap B(0,R)| \geq |\Omega| - \frac{\delta}{2}\,, \qquad \mathscr{H}^{N-1}_f\big(\partial\Omega \cap B(0,R)\big) \geq P(\Omega) - \frac{\varepsilon}{8}\,, \tag{2.2}$$

where for any $k > 0$ we define the measure $\mathscr{H}^k_f$ as $\mathscr{H}^k_f(A) = \int_A f(x)\, d\mathscr{H}^k(x)$ for every Borel set A.

Let us now consider the set Ω_j; recalling that

$$\mathscr{H}^{N-1}_f\big(\partial\Omega \cap B(0,R)\big) \leq \liminf_{j\to\infty} \mathscr{H}^{N-1}_f\big(\partial\Omega_j \cap B(0,R)\big)\,,$$

by (2.2) we immediately get that, if j is big enough, then

$$|\Omega_j| \leq |\Omega| + \delta\,, \quad |\Omega_j \cap B(0,R)| \geq |\Omega| - \delta\,, \quad \mathscr{H}_f^{N-1}\big(\partial\Omega_j \cap B(0,R)\big) \geq P(\Omega) - \frac{\varepsilon}{6}\,. \tag{2.3}$$

It is now possible to select some $R_j > R$ such that

$$\mathscr{H}_f^{N-1}\big(\Omega_j \cap S_{R_j}\big) \leq \frac{\varepsilon}{8}\,,$$

where for every $r > 0$ we denote by $S_r = \partial B(0,r)$ the sphere centered at the origin with radius r. We have then, calling $\Omega_j^- = \Omega_j \cap B(0,R_j)$ and $\Omega_j^+ = \Omega_j \setminus B(0,R_j)$,

$$P\big(\Omega_j^+\big) + P(\Omega_j^-) = P(\Omega_j) + 2\mathscr{H}_f^{N-1}\big(\Omega_j \cap S_{R_j}\big) \leq P(\Omega_j) + \frac{\varepsilon}{4}\,. \tag{2.4}$$

Notice now that, since $R_j > R$ and by (2.3), then

$$t_j := |\Omega| - |\Omega_j^-| \in [-\delta, \delta]\,,$$

thus we can define the competitor $\widetilde{\Omega}_j = \Omega_j^+ \cup F_{t_j}$, being the sets F_t as above. By construction, we have that $|\widetilde{\Omega}_j| = V$ for any $j \gg 1$, hence we have now to estimate the perimeters of $\widetilde{\Omega}_j$. Using that $R < R_j$ and (2.1), (2.4) and (2.3), we find that

$$\begin{aligned} P\big(\widetilde{\Omega}_j\big) &= P(F_{t_j}) + P(\Omega_j^+) \leq P(\Omega) - \frac{\varepsilon}{2} + P(\Omega_j^+) + P(\Omega_j^-) - P(\Omega_j^-) \\ &\leq P(\Omega) - \frac{\varepsilon}{2} + P(\Omega_j) + \frac{\varepsilon}{4} - \mathscr{H}_f^{N-1}\big(\partial\Omega_j \cap B(0,R)\big) \leq P(\Omega_j) - \frac{\varepsilon}{12}\,. \end{aligned}$$

Since this is in contrast with the fact that the original sequence $\{\Omega_j\}$ is isoperimetric, we have found an absurd, and this concludes the proof. □

The second result is a refinement of the first one, valid in the case when the density f converges to a limit at infinity. Here, and in the following, we will denote by $\mathfrak{J}(V)$ the infimum of the perimeters of sets of volume V (hence, if $\{\Omega_j\}$ is an isoperimetric sequence relative to volume V, then $P(\Omega_j) \to \mathfrak{J}(V)$).

Lemma 2.2. *In the same assumptions as in Lemma 2.1, if we further assume that $f \to 1$ at infinity, then*

$$\mathfrak{J}(V) = P(\Omega) + N\omega_N^{1/N}\big(V - |\Omega|\big)^{\frac{N-1}{N}}\,. \tag{2.5}$$

Proof. First of all, by approximation we can find a bounded set $\widetilde{\Omega}$ with

$$|\widetilde{\Omega}| = |\Omega|\,, \qquad\qquad P(\widetilde{\Omega}) \leq P(\Omega) + \varepsilon\,.$$

Then, consider a ball B of volume $V-|\Omega|$ very far from the origin, so that it does not intersect $\widetilde{\Omega}$: since $f\to 1$, we can assume that $P(B)$ is as close as we wish to the perimeter of a ball of volume $V-|\Omega|$ in the standard Euclidean space, which is $N\omega_N^{1/N}(V-|\Omega|)^{\frac{N-1}{N}}$: then, the set $\widetilde{\Omega}\cup B$ has exactly volume V and its perimeter is less than

$$P(\Omega)+N\omega_N^{1/N}\big(V-|\Omega|\big)^{\frac{N-1}{N}}+2\varepsilon\,;$$

since ε is arbitrary, this implies the first inequality in (2.5).

To obtain the other one, we can argue more or less as in the proof of the preceding lemma: having fixed a small constant $\varepsilon>0$, for every $j\gg 1$ we select R_j very big and such that

$$\left\{\begin{array}{c}\mathscr{H}_f^{N-1}(\Omega_j\cap S_{R_j})\leq\varepsilon\,,\\ \Big||\Omega_j\cap B(0,R_j)|-|\Omega|\Big|<\varepsilon\,,\\ P\big(\Omega_j\cap B(0,R_j)\big)>P(\Omega)-\varepsilon\,.\end{array}\right.$$

Since $f\to 1$, if R_j is big enough then the perimeter of $\Omega_j\setminus B(0,R_j)$ is arbitrarily close to the Euclidean perimeter of the same set, which is bigger than the Euclidean perimeter of the ball with the same volume: we deduce that

$$P\big(\Omega_j\setminus B(0,R_j)\big)\geq N\omega_N^{1/N}\big(V-|\Omega|\big)^{\frac{N-1}{N}}-\tilde{\varepsilon}\,,$$

for some $\tilde{\varepsilon}$ which depends on ε and goes to 0 when ε goes to 0. As a consequence, we get

$$\begin{aligned}P(\Omega_j)&=P\big(\Omega_j\cap B(0,R_j)\big)+P\big(\Omega_j\setminus B(0,R_j)\big)-2\mathscr{H}_f^{N-1}(\Omega_j\cap S_{R_j})\\&\geq P(\Omega)+N\omega_N^{1/N}\big(V-|\Omega|\big)^{\frac{N-1}{N}}-3\varepsilon-\tilde{\varepsilon}\,.\end{aligned}$$

Again recalling the ε is arbitrary, we obtain the other inequality and the proof is concluded. $\square$

Let us now give a simple definition; to introduce it, consider a ball B in a region where f is constantly C. As already noticed at the beginning of Section 1.2, we can immediately observe that

$$P(B)=C^{1/N}N\omega_N^{1/N}|B|^{\frac{N-1}{N}}\,.$$

We give then the following definition.

Definition 2.3. *We say that the* mean density *of the set* Ω *is the (unique) number* ρ *such that*

$$P(\Omega) = \rho^{1/N} N \omega_N^{1/N} |\Omega|^{\frac{N-1}{N}}\,.$$

Basically, the mean density of a set is the constant ρ such that, in a region where the density is constantly ρ, balls with the same volume as Ω have also the same perimeter. This definition, which first appeared in [5], could seem strange at first glance; nevertheless, for densities which converge to a limit, this turns out to be extremely useful thanks to the following result.

Lemma 2.4. *Let* f *be a density which converges to* 1 *at infinity. Assume that there exist bounded isoperimetric sets for any volume, and that there exist sets of any volume arbitrarily far from the origin and with mean density less than* 1*. Then, for every* $V > 0$ *there exists an isoperimetric set of volume* V*.*

Proof. Let $\{\Omega_j\}$ be an isoperimetric sequence corresponding to the volume V, and assume that Ω_j weakly converges to some Ω (this is always true, up to a subsequence). By Lemma 2.1 and Lemma 2.2 we know that Ω is an isoperimetric set for volume $|\Omega|$, and that (2.5) holds. By assumption, there exists a set E with volume $|E| = |\Omega|$ which is isoperimetric and also bounded. Again by assumption, there exists a set F with volume $V - |\Omega|$ which has mean density less than 1; since F can be taken arbitrarily far from the origin, we can assume that $E \cap F = \emptyset$.

The set $E \cup F$ is then a set with volume exactly V, and recalling that E is isoperimetric and that the mean density of F is less than 1, we get

$$P(E \cup F) \leq P(E) + P(F) = P(\Omega) + P(F) \leq P(\Omega) + N\omega_N^{1/N} |F|^{\frac{N-1}{N}} = \mathfrak{J}(V)\,.$$

Since this implies that $E \cup F$ is an isoperimetric set of volume V, we have concluded the proof. □

One of the assumptions of the above lemma is the *a priori* boundedness of isoperimetric sets, which has been recently well studied. In fact, it is always true under our assumptions, thanks to the following result, which concludes this section.

Lemma 2.5. *Assume that the density* f *is either continuous and bounded above and below far from the origin, or it converges to a positive limit at infinity. Then, all the isoperimetric sets are bounded.*

We do not give the proof of this result here, since it is quite involved. The proof of the continuous case can be found in [2], while the proof of the second case comes from an observation by F. Morgan, and can be found in [3].

3 Proof of our main result

This section is devoted to describe in detail the proof of Theorem 1.4, with a big emphasis on the underlying ideas.

Let us start by considering the claim of Lemma 2.4: thanks to Lemma 2.5, it tells us that to prove our Theorem 1.4 we can limit ourselves in finding sets with any volume and mean density less than 1 arbitrarily far from the origin (up to a trivial rescaling of f, we can of course assume without loss of generality that $\ell = 1$).

Observe that, in order to have a small mean density, a set should be suitably placed with respect to the density (the best would be if the density is big inside the set and small on its boudary), but it should also have a small perimeter in the Euclidean sense: if the density on the boundary is small, but the boundary has a huge extension, then this is not convenient... In particular, since we are looking for sets which are far from the origin, and the density converges to 1, the volume and perimeter are very close to the Euclidean volume and to the Euclidean perimeter; hence, a set with a mean density smaller than 1 must be very similar to a ball. For this reason, we start (in the first step) to search a ball with mean density smaller than one, and for simplicity we work in the simpler radial case. Then, in the second step we conclude the thesis of Theorem 1.4, still in the radial case. Having the strategy in mind, in steps 3 and 4 we do the same thing in the more complicate general case.

3.1 Step 1: A "good" ball of given radius when f is radial.

In this first step, we assume that f is radial (at least far from the origin), and we look for a ball of mean density less than 1 arbitrarily far from the origin. Notice that we must find such balls for any given volume. Up to a dilation, we can clearly reduce ourselves to consider only a particular value of the volume; however, it is important to underline immediately that we must choose this value *in advance.* In other words, just finding a ball with mean density less than 1 and a random volume is not sufficient! Even if this might seem not such a big problem, at first glance, this will nevertheless give some difficulties later: indeed, it is much easier (and this is what we will do) to search for a "good" ball (that is, a ball with mean density less than 1) having fixed its radius, not its perimeter. But then, we have to take care of adjusting the volume (in principle, it would be possible that there exist balls with mean density less than one with any possible radius, but not with any possible volume!).

For simplicity, let us then search for a good ball with radius 1: the goal of this first step is to prove the following result.

Lemma 3.1. *Under the assumptions of Theorem 1.4, if in addition f is radial then there exist balls of radius 1 and mean density less than 1 arbitrarily far from the origin.*

Notice that a ball very far from the origin and with unit radius has volume very close to ω_N, and perimeter very close to $N\omega_N$, since $f \to 1$; hence, by the definition, the mean density is very close to 1, so to check that some particular ball has mean density less than 1 we need a careful analysis of the difference between f and 1. Hence, it is useful to define the auxiliary density $g = 1 - f$, which by assumption is radial and positive far from the origin, and to evaluate volumes and perimeters also in terms of g: to avoid confusion, let us write then $P_g(E)$ and $|E|_g$ to denote the perimeter and volume of the set E with respect to the density g. Of course, any ball $B(R)$ of radius 1 and centered at distance R from the origin has volume and perimeter

$$|B(R)| = \omega_N - |B(R)|_g\,, \qquad P(B(x)) = N\omega_N - P_g(B(R))\,. \tag{3.1}$$

In order to prove Lemma 3.1, we first need a couple of simple elementary properties.

Lemma 3.2. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be definitively positive and converging to 0, and let $\alpha : (-1,1) \to \mathbb{R}$ be an L^1 function such that*

$$\int_{-1}^{1} \alpha(t)\,dt = 0\,, \qquad \int_{-1}^{\sigma} \alpha(t)\,dt > 0 \quad \forall\,\sigma \in (-1,1)\,. \tag{3.2}$$

Then, there exists R arbitrarily big such that

$$\int_{-1}^{1} \alpha(t) g(t+R)\,dt \geq 0\,,$$

with strict inequality unless $g \equiv 0$ in $(R-1, R+1)$.

Proof. Let us assume that the claim is false. Hence, for any $R^+ \gg R^- \gg 1$ we have

$$0 \geq \int_{R=R^-}^{R^+} \int_{t=-1}^{1} \alpha(t) g(t+R)\, dt\, dR = \int_{x=R^- -1}^{R^+ +1} g(x) \int_{A_x} \alpha(t)\, dt\, dx\,,$$

where A_x is defined by

$$A_x = \big\{ t \in (-1,1) : \, x - R^+ < t < x - R^- \big\}\,.$$

Notice now that, if $R^- + 1 < x < R^+ - 1$, then $A_x = (-1,1)$; as a consequence, by (3.2), for those x one has $\int_{A_x} \alpha(t)\, dt = 0$. So, the estimate above can be rewritten as

$$\begin{aligned} 0 &\geq \int_{x=R^- -1}^{R^- +1} g(x) \int_{A_x} \alpha(t)\, dt\, dx + \int_{x=R^+ -1}^{R^+ +1} g(x) \int_{A_x} \alpha(t)\, dt\, dx \\ &= \int_{x=R^- -1}^{R^- +1} g(x) \int_{-1}^{x-R^-} \alpha(t)\, dt\, dx + \int_{x=R^+ -1}^{R^+ +1} g(x) \int_{-1}^{x-R^+} \alpha(t)\, dt\, dx\,. \end{aligned}$$

Again by (3.2) we know that the first integral is positive and the second one is negative; moreover, if we keep R^- fixed and we send $R^+ \to \infty$, then the second integral converges to 0. We have to divide now two possibilities: if $g \equiv 0$ in $(R^- - 1, R^- + 1)$, then we have already the claim with $R = R^-$; otherwise, the first integral is *strictly* positive, hence for R^+ big enough we find a contradiction. In both cases, the proof is concluded. □

Lemma 3.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be definitively positive and converging to 0, and let $\alpha : (-1,1) \to \mathbb{R}$ be an L^1 function such that $\tilde{\alpha}(t) := \int_{-1}^{t} \alpha(\sigma)\, d\sigma$ satisfies the assumption of Lemma 3.2 as well as $\tilde{\alpha}(1) = 0$. Then there exists R arbitrarily big such that*

$$\int_{-1}^{1} \alpha(t) g(t+R)\, dt \geq 0\,.$$

Proof. We can argue exactly as in the previous Lemma. In fact, Let $R^+ \gg R^- \gg 1$ and assume that the claim is false for every $R^- \leq R \leq R^+$: since $\tilde{\alpha}(1) = 0$ means that $\int_{-1}^{1} \alpha(\sigma)\, d\sigma = 0$, with the same calculation as in the previous proof we can evaluate

$$\begin{aligned} 0 &> \int_{R=R^-}^{R^+} \int_{t=-1}^{1} \alpha(t) g(t+R)\, dt\, dR \\ &= \int_{x=R^- -1}^{R^- +1} g(x) \int_{-1}^{x-R^-} \alpha(t)\, dt\, dx + \int_{x=R^+ -1}^{R^+ +1} g(x) \int_{-1}^{x-R^+} \alpha(t)\, dt\, dx\,. \end{aligned}$$

Since second term is again going to 0 for $R^+ \to \infty$, to conclude we just need to prove that the first term is strictly positive. Let us rewrite it as

$$\begin{aligned}\int_{x=R^--1}^{R^-+1} g(x) \int_{-1}^{x-R^-} \alpha(t)\,dt\,dx &= \int_{x=R^--1}^{R^-+1} g(x)\tilde{\alpha}(x-R^-)\,dx \\ &= \int_{t=-1}^{1} g(t+R^-)\tilde{\alpha}(t)\,dt\,.\end{aligned}$$

Applying Lemma 3.2 to the function $\tilde{\alpha}$, we precisely obtain a suitable choice of R^- such that the above integral is strictly positive (our assumption reules out the possibility that $g \equiv 0$ in $(R^- - 1, R^- + 1)$), hence this proof is concluded. □

We are now in position to prove Lemma 3.1; as said above, it is notationally simpler to use the perimeters and volumes with respect to g, and then deduce the desired results for f.

Lemma 3.4. *Under the assumptions of Theorem 1.4, if in addition f is radial then for every $\varepsilon > 0$ there exists a ball $B(R)$ of radius 1 arbitrarily far from the origin such that*

$$P_g(B(R)) \geq (N-\varepsilon)|B(R)|_g\,. \tag{3.3}$$

Proof (of Lemma 3.1). The claim of Lemma 3.1 is a straightforward consequence of Lemma 3.4. Indeed, keeping in mind formulas (3.1), we know that the ball $B(R)$ has mean density less than 1 if and only if

$$N\omega_N - P_g(B(R)) < N\omega_N^{1/N}\Big(\omega_N - |B(R)|_g\Big)^{\frac{N-1}{N}}\,,$$

and since $g \to 0$ at infinity the previous inequality reduces to

$$P_g(B(R)) > \big(N-1+o(1)\big)|B(R)|_g\,,$$

where $o(1)$ is a quantity which goes to 0 when $|B(R)|_g$ goes to 0, hence when $R \to \infty$. Thus, the claim of Lemma 3.4 is stronger than what we need. □

Proof (of Lemma 3.4). Since the density is radial, we can express the perimeter and volume of $B(R)$ as

$$\begin{aligned}P_g(B(R)) &= \int_{x=R-1}^{R+1} \varphi_1(x-R,R)g(x)\,dx\,,\\ |B(R)| &= \int_{x=R-1}^{R+1} \varphi_2(x-R,R)g(x)\,dx\,,\end{aligned} \tag{3.4}$$

and the functions $\varphi_{1,2} : (-1,1) \times \mathbb{R}^+ \to \mathbb{R}^+$ clearly converge, for $R \to \infty$, to the functions

$$\overline{\varphi}_1(t) = (N-1)\omega_{N-1}(1-t^2)^{\frac{N-3}{2}}\,, \qquad \overline{\varphi}_2(t) = \omega_{N-1}(1-t^2)^{\frac{N-1}{2}}\,,$$

which depend only on t. Notice that the convergence is quite strong, namely the ratio $\varphi_{1,\,2}/\overline{\varphi}_{1,\,2}$ converge uniformly to one: as a consequence, the claim follows if we can find balls of radius 1 arbitrarily far from the origin such that

$$\overline{P}_g(B(R)) \geq N\overline{V}_g(B(R))\,, \tag{3.5}$$

where the modified perimeter and volume $\overline{P}_g$ and $\overline{V}_g$ are defined as

$$\overline{P}_g(B(R)) = \int_{t=-1}^{1} \overline{\varphi}_1(t)g(t+R)\,dx\,, \quad \overline{V}_g(B(R)) = \int_{t=-1}^{1} \overline{\varphi}_2(t)g(t+R)\,dx\,,$$

compare with (3.4). A trivial check ensures that the function $\alpha(t) = \overline{\varphi}_1(t) - N\overline{\varphi}_2(t)$ satisfies the assumptions of Lemma 3.3, and then the existence of some arbitrarily big R satisfying (3.5) follows, thus the proof is concluded. □

3.2 Step 2: Conclusion when f radial.

As described above, the proof of Theorem 1.4 follows as soon as we find a set of any given volume arbitrarily far from the origin. Since the density converges to 1, let us look for a set of volume ω_N, which is the volume of a ball of radius 1 at infinity. Lemma 3.1 of Step 1 already gives us balls arbitrarily far from the origin with radius 1 and mean density smaller than 1; however, since f is converging to 1 from below, the volume of these balls is slightly smaller than 1. As a consequence, we need to enlarge these balls a little. Notice that considering big balls would not be a good idea: indeed, since f is not even continuous, also a small movement of the boundary would possibly increase the perimeter too much, destroying the information about the mean density. Let us give here the construction.

Lemma 3.5. *Under the assumptions of Theorem 1.4, if in addition f is radial then there exists a set of volume ω_N and mean density smaller than* 1 *arbitrarily far from the origin.*

Proof. Thanks to the first step, in particular by (3.3) of Lemma 3.4, we get a ball of radius 1 and such that

$$P_g(B(R)) \geq (N-\varepsilon)|B(R)|_g\,. \tag{3.6}$$

Since $f \leq 1$, we know that $|B(R)| \leq \omega_N$, in particular by definition

$$|B(R)| = \omega_N - |B(R)|_g \,. \tag{3.7}$$

Hence, we enlarge the ball as shown in Figure 1. More precisely, we divide the ball $B(R)$ into two half-balls, the "upper one" and the "lower one", which are the two parts divided by an hyperplane passing trough the origin and the center of $B(R)$. Then, we "rotate down" the lower part of the boundary $\partial B(R)$, call it Σ^-, in such a way that it becomes a half-sphere $\widetilde{\Sigma}^-$ whose center is below the original one of a distance δ. Finally, we define the new set $\widetilde{B}$ as the set whose boundary is the union of the upper half-sphere $\partial B(R) \setminus \Sigma^-$, plus the new half-sphere $\widetilde{\Sigma}^-$, plus the union Γ of all the arcs of circle centered at the origin and connecting each point of Σ^- with the corresponding point of $\widetilde{\Sigma}^-$ (for the case of dimension $N = 2$, Γ is actually made only by two arcs of circle). In Figure 1 the new boundary $\partial\widetilde{B} \setminus \partial B(R)$ is made in dash.

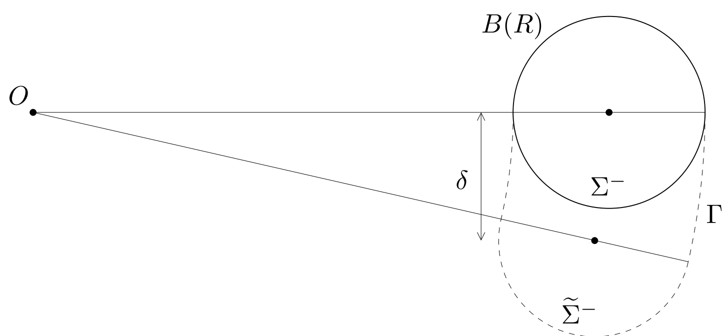

Figure 1: Ball expansion in Step 2.

The quantity δ is chosen in such a way that $|\widetilde{B}| = \omega_N$: as a consequence, we deduce that $\delta \ll 1$ and that $\widetilde{B} \supseteq B(R)$. Moreover, since the Euclidean volume of $\widetilde{B} \setminus B(R)$ is of course

$$|\widetilde{B} \setminus B(R)|_{\rm eucl} = \omega_{N-1}\delta(1 + o(1)) \,,$$

where $o(1)$ is a quantity which goes to 0 when $R \to \infty$, since $f \approx 1$, and since by construction –just keeping in mind (3.7)– we have $|\widetilde{B} \setminus B(R)| = |B(R)|_g$, we obtain

$$\delta = \frac{|B(R)|_g}{\omega_{N-1}} \,(1 + o(1)) \,.$$

We are now in position to calculate $P(\widetilde{B})$: since by construction, and thanks to the fact that f is radial, we have that $\mathscr{H}_f^{N-1}(\Sigma^-) = \mathscr{H}_f^{N-1}(\widetilde{\Sigma}^-)$, recalling (3.6) we directly have

$$\begin{aligned}
P(\widetilde{B}) &= P(B(R)) + \mathscr{H}_f^{N-1}(\Gamma) \leq P(B(R)) + \mathscr{H}^{N-1}(\Gamma) \\
&= P(B(R)) + (N-1)\omega_{N-1}\delta(1+o(1)) \\
&= P(B(R)) + (N-1)|B(R)|_g(1+o(1)) \leq P(B(R)) + P_g(B(R)) = N\omega_N \,,
\end{aligned}$$

where the last equality comes from the fact that $f+g=1$ and so $P(B(R)) + P_g(B(R))$ is the Euclidean volume of $B(R)$, which is a ball of radius 1. We conclude by noticing that the set $\widetilde{B}$ has volume ω_N by construction, and then having perimeter less than $N\omega_N$ is equivalent to have mean density less than one, which concludes the proof. □

3.3 Step 3: A "good" ball of given radius for a generic f.

In this and in the next step we are going to obtain the very same results as in Steps 1 and 2, but for a generic density f instead of a radial one. The result of this step is then the following result, which is analogous to Lemma 3.4: the idea of the proof is simply to use an auxiliary radial density, obtained by radially averaging f.

Lemma 3.6. *Under the assumptions of Theorem 1.4, there exists a ball $B_\beta(R)$ of radius 1 arbitrarily far from the origin such that*

$$P_g(B_\beta(R)) \geq (N-\varepsilon)|B_\beta(R)|_g \,. \tag{3.8}$$

Proof. Let us use polar coordinates, denoting every point $x \in \mathbb{R}^N$ as $x \equiv (\rho, \theta)$, where $\rho \geq 0$, $\theta \in \mathbb{S}^{N-1}$. Let us then define the radial density $\tilde{g}(\rho,\theta) = \tilde{g}(\rho)$, where

$$\tilde{g}(\rho) = \fint_{\mathbb{S}^{N-1}} g(\rho,\theta)\, d\mathscr{H}^{N-1}(\theta)\,,$$

that is, $\tilde{g}$ is the radial average of g. Now, fix a large $R \gg 1$: in the previous steps we simply called $B(R)$ any ball of unit radius with center at distance R from the origin, because all such balls were equivalent due to the radial assumption on the density. Since now f (thus $g = 1-f$) is generic, all the balls of unit radius and distance R from the origin may have different perimeters and different volumes, hence it is more convenient to call each of these balls $B_\beta(R)$ for $\beta \in \mathbb{S}^{N-1}$, with the obvious meaning that we fix arbitrarily one of these balls and then $B_\beta(R)$ is the ball obtained after a rotation of angle β. A trivial calculation then gives that, if we define $\Gamma(R,\rho)\subseteq\mathbb{S}^{N-1}$, for every $\rho\in(R-1,R+1)$, as the set

$$\Gamma(R,\rho)=\Big\{\theta\in\mathbb{S}^{N-1}:\,x\equiv(\rho,\theta)\in B_0(R)\Big\}\,,$$

then the g-volume of each ball $B_\beta(R)$ is

$$|B_\beta(R)|_g=\int_{\rho=R-1}^{R+1}\int_{\theta\in\Gamma(R,\rho)}g(\rho,\theta+\beta)\,d\mathscr{H}^{N-1}(\theta)\,d\rho\,.$$

As a consequence, we simply have

$$\begin{aligned}
&\fint_{\beta\in\mathbb{S}^{N-1}}|B_\beta(R)|_g\,d\mathscr{H}^{N-1}(\beta)\\
&\qquad=\fint_{\beta\in\mathbb{S}^{N-1}}\int_{\rho=R-1}^{R+1}\int_{\theta\in\Gamma(R,\rho)}g(\rho,\theta+\beta)\,d\mathscr{H}^{N-1}(\theta)\,d\rho\,d\mathscr{H}^{N-1}(\beta)\\
&\qquad=\int_{\theta\in\Gamma(R,\rho)}\int_{\rho=R-1}^{R+1}\fint_{\beta\in\mathbb{S}^{N-1}}g(\rho,\theta+\beta)\,d\mathscr{H}^{N-1}(\beta)\,d\rho\,d\mathscr{H}^{N-1}(\theta)\\
&\qquad=\int_{\theta\in\Gamma(R,\rho)}\int_{\rho=R-1}^{R+1}\tilde g(\rho)\,d\mathscr{H}^{N-1}(\beta)\,d\rho=|B(R)|_{\tilde g}\,,
\end{aligned}$$

where $B(R)$ is again the generic ball with distance R from the origin, since $\tilde g$ is radial. The very same calculation of course gives

$$\fint_{\beta\in\mathbb{S}^{N-1}}P_g(B_\beta(R))\,d\beta=P_{\tilde g}(B(R))\,. \tag{3.9}$$

Putting together the last two estimates we obtain that

$$\fint_{\beta\in\mathbb{S}^{N-1}}P_g(B_\beta(R))-(N-\varepsilon)|B_\beta(R)|_g\,d\beta=P_{\tilde g}(B(R))-(N-\varepsilon)|B(R)|_{\tilde g}\,,$$

thus there exists some $\beta\in\mathbb{S}^{N-1}$ such that

$$P_g(B_\beta(R))-(N-\varepsilon)|B_\beta(R)|_g\geq P_{\tilde g}(B(R))-(N-\varepsilon)|B(R)|_{\tilde g}\,. \tag{3.10}$$

We can now apply Lemma 3.4 to the density $\tilde f=1-\tilde g$, finding an arbitrarily big R such that

$$P_{\tilde g}(B(R))\geq(N-\varepsilon)|B(R)|_{\tilde g}\,,$$

hence by (3.10) the ball $B_\beta(R)$ satisfies (3.8). □

3.4 Step 4: Conclusion for a generic f.

In this last step we need then to generalize Lemma 3.5 removing the radial assumption on f. Keep in mind that the proof of Lemma 3.5 used this assumption twice; once, to apply Lemma 3.4, which was valid only for radial f but which has been generalized to non necessarily radial densities in Lemma 3.6. And once, in a crucial way, to know that $\mathscr{H}_f^{N-1}(\Sigma^-) = \mathscr{H}_f^{N-1}(\widetilde{\Sigma}^-)$, that is, all the half-spheres having the same distance from the origin have the same $\mathscr{H}^{N-1}$ measure. Since this last fact is clearly in general false for a non-radial density, we cannot simply take the ball $B_\beta(R)$ provided by Lemma 3.6 and modify it: indeed, it may happen that g is much bigger in the boundary of $B_\beta(R)$ than in the points nearby, and as a consequence the mean densities of even small adjustments of $B_\beta(R)$ could be strictly bigger than one. And in fact, in the argument of this last step we will not use the claim of Lemma 3.6, but a modification of its proof. The goal of this last step is to prove the following lemma, which as discussed above will conclude the proof of Theorem 1.4.

Lemma 3.7. *Under the assumptions of Theorem 1.4, there exists a set of volume ω_N and mean density smaller than 1 arbitrarily far from the origin.*

Proof. Let us present the proof for the planar case $N = 2$; in fact, it is much simpler to follow the construction, and to obtain the complete proof then just a simple linear algebra argument is needed, but no new idea.

First of all, let us define $\tilde{f}$ and $\tilde{g}$ the radial averages of f and g, as in the proof of Lemma 3.6; applying Lemma 3.4 to $\tilde{f}$, then, we find an arbitrarily large R such that

$$P_{\tilde{g}}(B) \geq (2-\varepsilon)|B|_{\tilde{g}}\,, \tag{3.11}$$

where for the sake of shortness we write B instead of $B(R)$, since R has been fixed once for the whole proof. Now, for every $\beta \in \mathbb{S}^1$, we call again B_β the ball of unit radius and centered at the point of polar coordinates $x \equiv (R, \beta)$. Moreover, as in Figure 2 we decompose $\partial B_\beta = \partial^+ B_\beta \cup \partial^- B_\beta$, where each point (x, θ) in ∂B_β belongs to $\partial^+ B_\beta$ (resp., $\partial^- B_\beta$) if $\theta \geq \beta$ (resp., $\theta \leq \beta$). Notice that since $R \gg 1$ then for every $(x, \theta) \in \partial B_\beta$ one has $\theta \approx \beta$, thus the above definition makes sense.

Notice also that, arguing exactly as in (3.9), we get

$$\begin{aligned} P_{\tilde g}(B) &= \fint_{\beta\in\mathbb{S}^{N-1}} P_g(B_\beta)\,d\beta = 2\fint_{\beta\in\mathbb{S}^{N-1}} \mathscr{H}^1_g(\partial^+ B_\beta)\,d\beta \\ &= 2\fint_{\beta\in\mathbb{S}^{N-1}} \mathscr{H}^1_g(\partial^- B_\beta)\,d\beta\,. \end{aligned} \tag{3.12}$$

Now, since $f\leq 1$, we have that $|B_\beta|\leq 1$, hence we have again to enlarge the

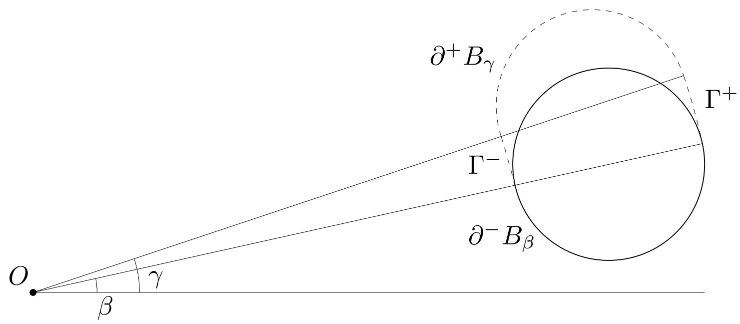

Figure 2: Situation in Step 4 and ball expansion.

ball; to do so, let us fix an angle β, and let $\gamma=\tau(\beta)$ to be specified in a moment. Then, let us call $\widetilde B_\beta$ the set whose boundary is the union of $\partial^- B_\beta$, $\partial^+ B_\gamma$, and two arcs of circle Γ^- and Γ^+ centered at the origin, with radii $R-1$ and $R+1$, and ranging from the direction β to the direction γ: see Figure 2 for a sketch of this set, where the dashed curves are $\partial\widetilde B_\beta\setminus\partial B_\beta$. The choice of γ is simple: we let $\gamma=\tau(\beta)$ be the angle such that $|\widetilde B_\beta|=\omega_N$. Notice that, since $f\approx 1$ because we are very far from the origin, there exists a unique such $\tau(\beta)$, and $\tau(\beta)-\beta\ll 1$.

Let us now take an angle β, and a very small $\delta\ll 1$: since $|B_\beta|=|B_{\beta+\delta}|$, then of course the volume of the "added part" $A^+_{\tau(\beta),\,\tau(\beta+\delta)}$ between $\partial^+ B_{\tau(\beta)}$ and $\partial^+ B_{\tau(\beta+\delta)}$ coincides with the volume of the "removed part" $A^-_{\beta,\beta+\delta}$ between $\partial^- B_\beta$ and $\partial^- B_{\beta+\delta}$. Since, up to take R big enough, we have $1-\varepsilon\leq f\leq 1$, then an immediate integration in polar coordinates ensures us that

$$2(1-\varepsilon)\delta(R-1)\leq \big|A^-_{\beta,\beta+\delta}\big|\leq 2\delta(R+1)\,,$$

and in the very same way

$$2(1-\varepsilon)\big(\tau(\beta+\delta)-\tau(\beta)\big)(R-1)\leq \big|A^+_{\tau(\beta),\tau(\beta+\delta)}\big|\leq 2\big(\tau(\beta+\delta)-\tau(\beta)\big)(R+1)\,. \tag{3.13}$$

Therefore, we deduce

$$(1-\varepsilon)\frac{R-1}{R+1}\leq \frac{\tau(\beta+\delta)-\tau(\beta)}{\delta}\leq \frac{R+1}{(R-1)(1-\varepsilon)}\,,$$

hence we obtain that $\tau : \mathbb{S}^1 \to \mathbb{S}^1$ is a Lipschitz function, and $\tau'(\beta) \in (1-2\varepsilon, 1+2\varepsilon)$, up to further increase R if necessary. We can then write, keeping in mind (3.12)

$$\begin{aligned}\int_{\beta=0}^{2\pi} \mathscr{H}^1_g\big(\partial^+(B_{\tau(\beta)}\big)\, d\beta &= \int_{\theta=0}^{2\pi} \frac{\mathscr{H}^1_g\big(\partial^+(B_\theta)\big)}{\tau'(\tau^{-1}(\theta)}\, d\theta \\ &\geq (1-2\varepsilon) \int_{\theta=0}^{2\pi} \mathscr{H}^1_g\big(\partial^+(B_\theta)\big)\, d\theta = \frac{1-2\varepsilon}{2}\, P_{\tilde g}(B)\,.\end{aligned}$$

As a consequence,

$$\int_{\beta=0}^{2\pi} \mathscr{H}^1_g\big(\partial^+(B_{\tau(\beta)}\big) + \mathscr{H}^1_g\big(\partial^-(B_\beta)\big)\, d\beta \geq (1-\varepsilon)\, P_{\tilde g}(B)\,.$$

Hence we can calculate, with the aid of (3.11),

$$\begin{aligned}\int_{\beta=0}^{2\pi} \big|B_\beta\big|_g\, d\beta = |B|_{\tilde g} &\leq \frac{P_{\tilde g}(B)}{2-\varepsilon} \\ &\leq \frac{1}{(2-\varepsilon)(1-\varepsilon)} \int_{\beta=0}^{2\pi} \mathscr{H}^1_g\big(\partial^+(B_{\tau(\beta)}\big) + \mathscr{H}^1_g\big(\partial^-(B_\beta)\big)\, d\beta\,,\end{aligned}$$

and this implies the existence of some $\beta \in \mathbb{S}^1$ such that

$$\mathscr{H}^1_g\big(\partial^+(B_{\tau(\beta)}\big) + \mathscr{H}^1_g\big(\partial^-(B_\beta)\big) \geq (2-3\varepsilon)\,\big|B_\beta\big|_g\,. \tag{3.14}$$

We finally claim that the set $\widetilde{B}_\beta$ has mean density less than 1: since $|\widetilde{B}_\beta| = \pi = \omega_2$ by construction, this will conclude the proof. To start, notice that the ball B_β has of course Euclidean volume equal to π, so its volume with respect to f is $\pi - |B_\beta|_g$; on the other hand, π is the volume with respect to f of the enlarged set $\widetilde{B}_\beta$, which coincides with the union of B_β with the "added part" $A^+_{\beta,\tau(\beta)}$. This implies that $|A^+_{\beta,\tau(\beta)}| = |B_\beta|_g$, which with the same argument as in (3.13) gives

$$\tau(\beta) - \beta \leq \frac{|B_\beta|_g}{2(1-\varepsilon)(R-1)}\,. \tag{3.15}$$

Notice now that the perimeter of $\widetilde{B}_\beta$ is the sum of the lengths of the two half-circles $\partial^- B_\beta$ and $\partial^+ B_{\tau(\beta)}$, plus the two arcs Γ^- and Γ^+ in Figure 2. And in turn, the lenghts of those two arcs are smaller than the Euclidean lengths (since $f \leq 1$), and the Euclidean lengths are $(R-1)(\tau(\beta)-\beta)$ and $(R+1)(\tau(\beta)-\beta)$ respectively.

Summarizing, by (3.15) and (3.14) one has

$$
\begin{aligned}
P(\widetilde{B}_\theta) &\leq \mathscr{H}^1_f(\partial^- B_\beta) + \mathscr{H}^1_f(\partial^+ B_{\tau(\beta)}) + 2R(\tau(\beta)-\beta) \\
&\leq 2\pi - \Big(\mathscr{H}^1_g(\partial^- B_\beta) + \mathscr{H}^1_g(\partial^+ B_{\tau(\beta)})\Big) + \frac{R|B_\beta|_g}{(1-\varepsilon)(R-1)} \\
&\leq 2\pi - (2-3\varepsilon)|B_\beta|_g + \frac{R|B_\beta|_g}{(1-\varepsilon)(R-1)} \leq 2\pi\,,
\end{aligned}
$$

where the last inequality is true as soon as ε and R have been chosen sufficiently small and sufficiently big respectively.

We have finally concluded, because the last inequality is equivalent to the fact that the mean density of $|\widetilde{B}_\beta|$) is smaller than one. □

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