LOCAL BOUNDEDNESS OF MINIMIZERS WITH LIMIT GROWTH CONDITIONS

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Abstract. The energy-integral of the calculus of variations (1.1), (1.2) below has a limit behavior when $q = n\overline{p}/(n-\overline{p})$, where $\overline{p}$ is the harmonic average of the exponents p_i, $i = 1, \dots, n$. In fact, if q is larger than in the stated equality, counterexamples to the local boundedness and regularity of minimizers are known. In this paper we prove the local boundedness of minimizers (and also of quasi-minimizers) under this stated limit condition. Some other general and limit growth conditions are also considered.

1. Introduction

We are interested in the local boundedness of minimizers of general integrals of the Calculus of Variations of the type

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, u, Du(x))\, dx, \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, satisfying some convexity and growth conditions. In particular, we consider $f(x, s, \xi)$ convex in (s, ξ) and satisfying growth conditions such as

$$c_1 \sum_{i=1}^n |\xi_i|^{p_i} \leq f(x, s, \xi) \leq c_2 \left\{1 + |s|^q + |\xi|^q\right\}, \tag{1.2}$$

for some $1 \leq p_i \leq q$ and $c_1, c_2 > 0$. It is known that the boundedness is not guaranteed if the exponents are too spread, see Giaquinta [18], Marcellini [24], Hong [21] for counterexamples. In this note we prove that the reverse limit condition

$$q \leq \frac{n\overline{p}}{n-\overline{p}}, \quad \text{where} \quad \frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}$$

is really sufficient to get locally bounded minimizers. We also consider another limit case, when at least one of the exponents p_i in (1.2) is equal to 1, by considering the extension of the functional to BV.

Some other generalities considered in this paper are described in more details below. Precisely, we are concerned more generally with *quasi-minimizers*. We say that $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R})$ is a *quasi-minimizer* of (1.1) if there exists $Q \geq 1$ such that

$$\begin{cases} \mathcal{F}(u; \text{supp } \varphi) < +\infty, \\ \mathcal{F}(u; \text{supp } \varphi) \leq Q\mathcal{F}(u + \varphi; \text{supp } \varphi), \end{cases}$$

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for all $\varphi \in W^{1,1}(\Omega)$ with $\operatorname{supp}\varphi \Subset \Omega$. If $Q = 1$, then u is a *local minimizer* of (1.1); the results presented in this paper apply and seem to be new also for local minimizers.

In the classical theory of regularity and in some recent developments the integrand f satisfies growth conditions depending on Du through its modulus $|Du|$, such as

$$c_1|\xi|^p \leq f(x,s,\xi) \leq c_2\left(1+|\xi|^q\right). \tag{1.3}$$

If $p = q$, then (1.3) is called $p-$growth or standard or natural growth. If $p < q$ we are in the framework of $p, q-$growth. The regularity of local minimizers under $p, q-$growth has been extensively studied in the last years starting by Marcellini [25], [26].

If in (1.2) the power $|\xi|^q$ in the right hand side is replaced by $\sum_{i=1}^n |\xi_i|^{p_i}$, so obtaining an estimate of the growth of f from above of the same type than from below, the local boundedness of local minimizers has been studied by Boccardo-Marcellini-Sbordone [3], Stroffolini [29] and by Fusco-Sbordone [16], [17].

In (1.2) we may have $\max\{p_i\} < q$, as for instance for the functional

$$\mathcal{F}_1(u) = \int_\Omega \left(\sum_{i=1}^n |u_{x_i}|^{p_i} + a(x)|Du|^q\right) dx,$$

whenever $a(x) \geq 0$ is measurable and $a(x) = 0$ on a set of positive measure. In this context, see the recent interesting papers by Colombo-Mingione [4], [5], and see also Esposito-Leonetti-Mingione [15].

The local boundedness of minimizers of functionals with anisotropic growth (1.2) has already been studied by the authors in [6], see also [7], [8] for the vector valued case and [9] for the existence and regularity of solutions to elliptic equations.

In the present paper, we adopt a different strategy than in [6], where the Euler equation and the Moser iteration scheme was used. Here we derive the local boundedness by the De Giorgi method of super-(sub-)level sets. This allows to improve the previous results in different directions. Firstly, the De Giorgi method permits to consider Carathéodory integrands f and to admit the dependence of f on s. Secondly, we obtain the boundedness for *quasi-minimizers* of $\mathcal{F}$ and not only for *local minimizers*. Notice that local minimizers of integral functionals with Carathéodory integrand $g(x,s,\xi)$, possibly neither convex nor regular, but bounded from below and above by $f(x,s,\xi)$, i.e,

$$m(f(x,s,\xi)-1) \leq g(x,s,\xi) \leq M(f(x,s,\xi)+1) \qquad \text{with } m, M > 0,$$

are quasi-minimizers of $\int_\Omega\{f(x,u,Du)+1\}\,dx$. Thus, our results apply also to local minimizers in this case. Thirdly, a more general assumption on q can be done. To discuss further this feature, we need to define the harmonic average $\overline{p}$ of $\{p_i\}$ and its Sobolev exponent $\overline{p}^*$:

$$\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^n \frac{1}{p_i} \qquad \text{and} \qquad \overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p} < n, \\ \text{any } \mu > \overline{p} & \text{if } \overline{p} \geq n. \end{cases}$$

We prove that if $q \leq \overline{p}^*$, then the quasi-minimizers of $\mathcal{F}$ are locally bounded. We are able to include the limit case $q = \overline{p}^*$, that the procedure of the Moser iteration argument was unable to include. Precisely, we prove that given a quasi-minimizer u, if $q = \overline{p}^*$ and if one of the following two assumptions holds:

$$\max\{p_i\} < \overline{p}^* \quad \text{or} \quad u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega),$$

then u is locally bounded. We emphasize that in the known literature for the case $q = \overline{p}^*$ the condition that u a-priori belongs to $L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$ is sometimes omitted. On the contrary, this condition is needed for the embedding results of the anisotropic Sobolev spaces. In fact, the natural space to consider minimizers of $\mathcal{F}$ is

$$W^{1,(p_1,\dots,p_n)}(\Omega) := \left\{u \in W^{1,1}(\Omega) \, : \, u_{x_i} \in L^{p_i}(\Omega), \, i = 1,\dots,n\right\}.$$

Let Ω be a rectangular domain with edges parallel to the coordinate axes. If $\max\{p_i\} < \overline{p}^*$, then $W^{1,(p_1,\dots,p_n)}(\Omega) \subset L^{\overline{p}^*}(\Omega)$, see Troisi [30] and Acerbi-Fusco [1]. Otherwise, if $\max\{p_i\} = \overline{p}^*$, then the embedding is not guaranteed; see Kruzhkov-Kolodii [22] and Haskovec-Schmeiser [20] for counterexamples, see also Remark 3.4 for further details.

We emphasize also that Theorems 2.2 and 2.3 actually cover functionals more general than (1.1), (1.2). More precisely the growth condition considered (see Assumption (H2) in Section 2) takes into account a function g satisfying the $\triangle_2$-property (see (2.3)). Related boundedness results are in Dall'Aglio-Mascolo-Papi [10], Mascolo-Papi [27] and Moscariello-Nania [28].

Moreover we study a class of variational integrals with linear growth from below; i.e., (1.2) with $\min\{p_i\} = 1$. Due to the lack of coerciveness, we consider the relaxed functional of $\mathcal{F}$ in the $BV(\Omega)$: fixed $u_0 \in W^{1,1}(\Omega)$ with $\int_\Omega f(x, u_0, Du_0) < +\infty$, the relaxed functional is

$$\overline{\mathcal{F}}(u) := \inf\left\{\liminf_{k\to+\infty} \mathcal{F}(u_k) \, : \, u_k \to u \text{ in } L^1(\Omega), \; u_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)\right\}, \qquad u \in BV(\Omega).$$

We prove that there exists a locally bounded minimizer $\bar{u} \in BV$ of $\overline{\mathcal{F}}$ and an estimate of the L^∞-norm is given, see Theorem 2.7. We refer to Beck-Schmidt [2] for related results for functionals with linear growth.

The use of De Giorgi method for local boundedness suggests to go on: under which conditions either the minimizers or the quasi-minimizers are Hölder continuous? This problem is at the same time appealing and difficult to be treated. Few recent tentative approaches for functionals and operators with special structures are available in the literature: see Liskevich-Skrypnik [23], Düzgün-Marcellini-Vespri [12], [13], Colombo-Mingione [4], [5].

The contents of the paper is described next briefly. In Section 2 we give the precise hypotheses and statements of the regularity results: Theorems 2.2 and 2.3 (cases $q < \overline{p}^*$ and $q = \overline{p}^*$, respectively) and Theorem 2.5 ($p_i = p$ for all i). We also state regularity results for minimizers of functionals in suitable Dirichlet classes, dealing both with the coercive case ($\min\{p_i\} > 1$), see Theorem 2.6, and the non-coercive case ($\min\{p_i\} = 1$), Theorem 2.7. In this last case a generalized definition of minimizers is used, e.g. minimizers in BV for the associated relaxed functional. Section 5 and Section 6 contain the proofs of Theorems 2.2, 2.3 and, respectively, of Theorem 2.7. The proofs rely on embedding results for anisotropic Sobolev spaces and on a suitable Caccioppoli inequality: these results can be found in Sections 3 and 4, respectively.

2. Assumptions and statement of the main results

Define the integral functional

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, u, Du(x))\, dx. \tag{2.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $u \in W^{1,1}(\Omega, \mathbb{R})$.

Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, such that

(H1) either f is convex in the pair (s, ξ)
or
f is separately convex in s and ξ and $\lim_{|s|\to+\infty} f(x, s, \xi) = +\infty$ uniformly w.r.t. x and ξ.

(H2) there exist $c_1, c_2 > 0$ and $1 \le p_i \le q$, $i = 1, \dots, n$, such that

$$c_1 \sum_{i=1}^{n} [g(|\xi_i|)]^{p_i} \le f(x, s, \xi) \le c_2 \{1 + [g(|s|)]^q + [g(|\xi|)]^q\} \tag{2.2}$$

for a.e. x and every $\xi \in \mathbb{R}^n$.

Here, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^1, convex, non-decreasing, $g(0) = 0$, $g \not\equiv 0$, satisfying for some $\mu \ge 1$

$$g(\lambda t) \le \lambda^\mu g(t) \quad \text{for every} \quad \lambda > 1 \quad \text{and every} \quad t \ge t_0. \tag{2.3}$$

Without loss of generality, we assume t_0 large so that $g(t) \ge 1$ for all $t \ge t_0$. Moreover, notice that if the second alternative in (H1) holds, then:

$$\exists\, M \ge 0 \text{ such that } f(x, \cdot, \xi) \text{ is decreasing in } (-\infty, -M] \text{ and increasing in } [M, +\infty). \tag{2.4}$$

In this case we can also assume $t_0 \ge 2M$.

Let us now give the definition of quasi-minimizers of (2.1).

Definition 2.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *quasi-minimizer* of (2.1) if there exists $Q \ge 1$ such that $\mathcal{F}(u; \mathrm{supp}\ \varphi) < +\infty$ and

$$\mathcal{F}(u; \mathrm{supp}\ \varphi) \le Q \mathcal{F}(u + \varphi; \mathrm{supp}\ \varphi),$$

for all $\varphi \in W^{1,1}(\Omega)$ with $\mathrm{supp}\, \varphi \Subset \Omega$. If $Q = 1$, then u is a *local minimizer* of (2.1).

It is well known that restrictions on the exponents $\{p_i\}$ and q are necessary to have the local boundedness of quasi-minimizers of (2.1). We denote by $\overline{p}$ the harmonic average of $\{p_i\}$, i.e., $\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}$, finally $\overline{p}^*$ is the Sobolev exponent of $\overline{p}$

$$\overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p} < n, \\ \text{any } s > \overline{p} & \text{if } \overline{p} \ge n. \end{cases} \tag{2.5}$$

Our first result deals with the case $q < \overline{p}^*$.

Theorem 2.2. *Assume (H1) and (H2). If $q < \overline{p}^*$, then any quasi-minimizer u of (2.1) is locally bounded. Moreover, fixed $B_R(x_0) \Subset \Omega$, there exists a constant c depending on $q, p_i, \mu, Q, t_0, c_1, c_2$, such that*

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le c \left\{ 1 + \frac{1}{R^{\mu \frac{q\overline{p}^*}{p(\overline{p}^*-q)}}} \left(\int_{B_R(x_0)} g^q(|u|)\, dx \right)^{\frac{1+\theta}{q}} \right\}, \tag{2.6}$$

where $\theta = \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$, with $p = \min\{p_i\}$.

As far as the borderline case $q = \overline{p}^*$ is concerned, we have the following result.

Theorem 2.3. *Assume (H1) and (H2). If $q = \overline{p}^*$ and*

$$\textit{either} \quad \max\{p_i\} < \overline{p}^* \quad \textit{or} \quad g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega),$$

then any quasi-minimizer u of (2.1) is locally bounded.

Example 2.4. *Let us consider the functional*

$$\mathcal{F}(u) = \int_\Omega \left(\sum_{i=1}^{n} |u_{x_i}|^{p_i} + a(x)|u_{x_n}|^q \right) dx,$$

with $1 \leq p_2 \leq \ldots \leq p_n$*. Assume that* $a \not\equiv 0$*, with* $a(x) = 0$ *on a set of positive measure: if* $p_n < q = \overline{p}^*$ *then the quasi-minimizers of* $\mathcal{F}$ *are locally bounded. Assume now* $a(x) \equiv 1$*: if* $p_n = q = \overline{p}^*$*, then we can conclude that any quasi-minimizer* $u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$ *of* $\mathcal{F}$ *is locally bounded.*

Notice that if the p_i's are equal, a straightforward consequence of the above results is the following.

Theorem 2.5. *Assume (H1) and*

$$c_1|\xi|^p \leq f(x,s,\xi) \leq c_2 \left\{1 + |s|^q + |\xi|^q\right\}.$$

If $1 \leq p < q \leq p^*$*, then the quasi-minimizers of* $\mathcal{F}$ *are locally bounded.*

Now, we deal with the minimization problem in a Dirichlet class. To do this, we consider $g(t) = t$; i.e.,

(H3) there exist $c_1, c_2 > 0$ and $1 \leq p_i \leq q$, $i = 1, \ldots, n$, such that

$$c_1 \sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x,s,\xi) \leq c_2 \left\{1 + |s|^q + |\xi|^q\right\}$$

for a.e. x and every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

A first result, with $\min\{p_i\} > 1$ is the following.

Theorem 2.6. *Assume (H1) and (H3) with* $1 < p_i \leq q \leq \overline{p}^*$*,* $i = 1, \ldots, n$*. Let* $u_0 \in W^{1,1}(\Omega) \cap L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$ *be such that* $\mathcal{F}(u_0; \Omega) < +\infty$*. If* u *is a minimizer of* $\mathcal{F}(\cdot; \Omega)$ *in* $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$*, then* u *is locally bounded.*

Now, let us consider the analogous of Theorem 2.6, under the assumption $\min\{p_i\} = 1$. Fixed $u_0 \in W^{1,1}(\Omega)$ such that $\mathcal{F}(u_0; \Omega) < +\infty$. Since $\min\{p_i\} = 1$, then $W^{1,(p_1,\ldots,p_n)}(\Omega)$ is a non-reflexive space and the direct method generally fails. So, minimizers of $\mathcal{F}$ in $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$ may not exist. We claim that minimizers in BV of the relaxed functional in $BV(\Omega)$ of $\mathcal{F}$, i.e.,

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{F}(u_k) \,:\, u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega) \right\},$$

exist and are locally bounded.

Theorem 2.7. *Assume (H1) and (H3) hold, with* $1 \leq p_i \leq q < \overline{p}^*$*,* $\min\{p_i\} = 1$*.*
Fixed $u_0 \in W^{1,1}(\Omega)$*, such that* $\mathcal{F}(u_0; \Omega) < +\infty$*, there exists a minimizer* $\bar{u} \in BV(\Omega)$ *of* $\overline{\mathcal{F}}$ *such that* $\bar{u} \in L^\infty_{\mathrm{loc}}(\Omega)$ *and, for all* $B_R(x_0) \Subset \Omega$*,*

$$\|\bar{u}\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq c \left\{ 1 + \frac{1}{R^{\mu \frac{q\overline{p}^*}{\overline{p}^* - q}}} \left(\overline{\mathcal{F}}(\bar{u}) + 1 + \|u_0\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)} \right)^{1+\theta} \right\},$$

where $\theta = \frac{\overline{p}^*(q-1)}{\overline{p}^* - q}$*.*

3. Anisotropic Sobolev spaces

To prove our results we use a suitable anisotropic Sobolev space. Precisely,

$$W^{1,(p_1,\dots,p_n)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \,:\, u_{x_i} \in L^{p_i}(\Omega), \text{ for all } i = 1,\dots,n \right\},$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

We write $W_0^{1,(p_1,\dots,p_n)}(\Omega)$ in place of $W_0^{1,1}(\Omega) \cap W^{1,(p_1,\dots,p_n)}(\Omega)$. Notice that in this last space an equivalent norm of u is given by $\sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}$.

We recall the following embedding results for anisotropic Sobolev spaces. We refer to [30] and [1].

Theorem 3.1. *Let $p_i \geq 1$, $i = 1,\dots,n$, and $\overline{p}^*$ be as in (2.5). If $u \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$, with Ω bounded open set in $\mathbb{R}^n$, then there exists c, depending on n, p_i and, only in the case $\overline{p} \geq n$, also on $\overline{p}^*$ and the measure of the support of u, such that*

$$\|u\|_{L^{\overline{p}^*}(\Omega)} \leq c \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

Theorem 3.2. *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes and consider $u \in W^{1,(p_1,\dots,p_n)}(Q)$, $p_i \geq 1$ for all $i = 1,\dots,n$. If $\overline{p} < n$ assume also that $\max\{p_i\} < \overline{p}^*$.*
Then $u \in L^{\overline{p}^*}(Q)$. Moreover, there exists c depending on n, p_i and, if $\overline{p} \geq n$, also on $\overline{p}^*$ and the measure of the support of u, such that

$$\|u\|_{L^{\overline{p}^*}(Q)} \leq c \left\{ \|u\|_{L^1(Q)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(Q)} \right\}. \tag{3.1}$$

We also need the following result, see Proposition 1 in [6].

Proposition 3.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and let g satisfy the assumptions described in Section 2. Suppose that $g(|u_{x_i}|) \in L^{p_i}_{loc}(\Omega)$ with $1 \leq p_i < \overline{p}^*$ for every $i = 1,\dots,n$. Then $g(|u|) \in L^{\overline{p}^*}_{loc}(\Omega)$.*

Remark 3.4. Let $n \geq 2$. In general the inclusion $W^{1,(p_1,\dots,p_n)}(\Omega) \subset L^{\overline{p}^*}(\Omega)$ does not hold even if Ω is a rectangular domain. Let assume $\overline{p} < n$, that is $\sum_{i=1}^{n} \frac{1}{p_i} > 1$, and, without loss of generality, assume $p_1 \leq p_2 \leq \ldots \leq p_n$. Define, for $k = 1, ..., n$,

$$q^k := \begin{cases} \dfrac{k}{\sum_{i=1}^{k} \frac{1}{p_i} - 1} & \text{if } \sum_{i=1}^{k} \frac{1}{p_i} > 1 \\ +\infty & \text{else.} \end{cases}$$

If $p_n = \overline{p}^*$, we have $q^{n-1} = q^n = \overline{p}^*$. Thus, by Lemma 1 and Theorem 6 in [20], $W^{1,(p_1,\dots,p_n)}(\Omega)$ is continuosly embedded into every $L^q(\Omega)$ with $q < \overline{p}^*$. In [20] it is also proved that if $q^{n-1} > q^n$, then $W^{1,(p_1,\dots,p_n)}(\Omega)$ is continuously embedded into $L^{\overline{p}^*}(\Omega)$.

4. Caccioppoli inequality

First of all, we recall some properties of the Δ_2-functions, see [6] for the proof.

Lemma 4.1. *Consider $g : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1, convex, non-decreasing and satisfying (2.3). Then*

$$g(\lambda t) \le \lambda^\mu (g(t) + g(t_0)) \quad \text{and} \quad g'(t)t \le \mu(g(t) + g(t_0)) \qquad \text{for all } t \ge 0 \text{ and all } \lambda > 1.$$

Moreover, for every $(t_1, \dots, t_k) \in \mathbb{R}^k_+$ we have:

$$k^{-1} \sum_{i=1}^k g(t_i) \le g\left(\sum_{i=1}^k t_i\right) \le k^\mu \left\{ g(t_0) + \sum_{i=1}^k g(t_i) \right\}.$$

Now, we give a lemma about the properties related to the convexity assumptions on function $f = f(x, s, \xi)$.

Lemma 4.2. *If the second alternative in (H1) holds, then for every $\xi_1, \xi_2 \in \mathbb{R}^n$ we have*

$$f(x, ts_1 + (1-t)s_2, t\xi_1 + (1-t)\xi_2) \le t^2 f(x, s_1, \xi_1) + (1-t) f(x, s_2, \xi_2) + t(1-t) f(x, s_2, \xi_1)$$

whenever $0 \le t \le 1$ and $M \le s_1 \le s_2$ or $s_2 \le s_1 \le -M$. Here M is as in (2.4).

Proof. Using the convexity of f in the second and in the third variable, we have

$$f(x, ts_1 + (1-t)s_2, t\xi_1 + (1-t)\xi_2) \le t^2 f(x, s_1, \xi_1) + t(1-t)\{f(x, s_1, \xi_2) + f(x, s_2, \xi_1)\} + (1-t)^2 f(x, s_2, \xi_2).$$

Since $f(x, s_1, \xi_2) \le f(x, s_2, \xi_2)$ the thesis follows. □

The following is a well known classical result, see e.g. [19].

Lemma 4.3. *Let $\phi(t)$ be a nonnegative bounded function, defined in $[\tau_0, \tau_1]$. Suppose that, for all s, t, such that $\tau_0 \le s < t \le \tau_1$, ϕ satisfies*

$$\phi(s) \le \theta \phi(t) + \frac{A}{(t-s)^\alpha} + B$$

where A, B, α are non-negative constants and $0 < \theta < 1$.
Then, for all ρ and R, such that $\tau_0 \le \rho \le R \le \tau_1$, we have

$$\phi(\rho) \le C \left\{ \frac{A}{(R-\rho)^\alpha} + B \right\}.$$

If $u \in W^{1,1}(\Omega)$ and $B_R(x_0) \subseteq \Omega$ is ball, we define the super-level sets:

$$A_{k,R} := \{x \in B_R(x_0) \,:\, u(x) > k\} \qquad k \in \mathbb{R}.$$

Then the following Caccioppoli inequality holds.

Theorem 4.4. *Assume (H1), (H2) and let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$ such that $g(|u|) \in L^q_{\text{loc}}(\Omega)$. Then there exists a constant $c > 0$, such that for any $B_R(x_0) \Subset \Omega$, $0 < \rho < R \le \rho + 1$ and for any k and d such that $\frac{t_0}{2} \le k \le d$,*

$$\int_{A_{k,\rho}} f(x, u, Du)\, dx \le \frac{c}{(R-\rho)^{\mu q}} \int_{A_{k,R}} \{g^q(u-k) + g^q(d)\}\, dx. \tag{4.1}$$

Proof. Let $B_R(x_0) \Subset \Omega$. Let ρ, s, t be such that $\rho \le s < t \le R \le \rho + 1$. Let $\eta \in C_0^\infty(B_t)$ be a cut-off function, satisfying the following assumptions:

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{4.2}$$

Fixed $k \in \mathbb{R}_+$, define

$$w = \max(u-k, 0), \quad \text{and} \quad \varphi = -\eta^{\mu q} w.$$

Consider a number d, such that $d \geq k$. By the quasi-minimality of u we get:

$$\int_{A_{k,s}} f(x,u,Du)\,dx \leq Q \int_{A_{k,t}} f(x,u+\varphi,Du+D\varphi)\,dx$$
$$= Q \int_{A_{k,t}} f\left(x,(1-\eta^{\mu q})u+\eta^{\mu q}k,(1-\eta^{\mu q})Du+\mu q\eta^{\mu q-1}(k-u)D\eta\right)\,dx$$

Case 1. Let us assume that the first alternative in (H1) holds.
If f is convex in (s,ξ), by (H2) we have that for a.e. $x \in \{\eta \neq 0\}$

$$f\left(x,(1-\eta^{\mu q})u+\eta^{\mu q}k,(1-\eta^{\mu q})Du+\eta^{\mu q}\left(\mu q\frac{k-u}{\eta}D\eta\right)\right)$$
$$\leq (1-\eta^{\mu q})f(x,u,Du)+\eta^{\mu q}f(x,k,\mu q\frac{k-u}{\eta}D\eta)$$
$$\leq (1-\eta^{\mu q})f(x,u,Du)+c_2\eta^{\mu q}\left\{1+g^q(\mu q|\frac{u-k}{\eta}D\eta|)+g^q(d)\right\}.$$

Lemma 4.1 and (4.2) imply

$$g(|\mu q\frac{u-k}{\eta}D\eta|) \leq \frac{(2\mu q)^\mu}{(t-s)^\mu\eta^\mu}\left\{g(|u-k|)+g(t_0)\right\}. \tag{4.3}$$

Taking into account that $\operatorname{supp}(1-\eta^{\mu q}) \subset A(k,t)\setminus A(k,s)$ and $t \leq R$, we obtain

$$\int_{A_{k,s}} f(x,u,Du)\,dx$$
$$\leq Q\int_{A_{k,t}}\left\{(1-\eta^{\mu q})f(x,u,Du)+\frac{c}{(t-s)^{\mu q}}\left(g^q\left(u-k\right)+g^q(t_0)\right)+g^q(d)+1\right\}\,dx$$
$$\leq Q\int_{A_{k,t}\setminus A_{k,s}} f(x,u,Du)\,dx+\frac{c_3}{(t-s)^{\mu q}}\int_{A_{k,R}}\left(g^q\left(u-k\right)+g^q(d)\right)\,dx \tag{4.4}$$

with $c_3 = c_3(n,\mu,q,Q,c_2)$.

Case 2. Let us assume that the second alternative in (H1) holds.
By Lemma 4.2 with $t = \eta^{\mu q}(x)$, $s_1 = k$, $s_2 = u(x)$, $\xi_1 = \mu q\frac{k-u}{\eta}D\eta$, $\xi_2 = Du(x)$, and using $k \geq M$, we get that for a.e. $x \in \{u \geq k\} \cap \{\eta \neq 0\}$

$$f\left(x,(1-\eta^{\mu q})u+\eta^{\mu q}k,(1-\eta^{\mu q})Du+\eta^{\mu q}\mu q\frac{k-u}{\eta}D\eta\right)$$
$$\leq (1-\eta^{\mu q})^2f(x,u,Du)+\eta^{2\mu q}f(x,k,\mu q\frac{k-u}{\eta}D\eta)+\eta^{\mu q}f(x,u,\mu q\frac{k-u}{\eta}D\eta).$$

Now, using (H2), $k \leq d$, and (4.3)

$$f(x,k,\mu q\frac{k-u}{\eta}D\eta) \leq c_2\left\{1+g^q(d)+g^q(\mu q\frac{|u-k|}{\eta}|D\eta|)\right\} \leq \frac{c}{(t-s)^{\mu q}\eta^{\mu q}}\{g^q(|u-k|)+g^q(d)\}.$$

Analogously, taking into account that in $A_{k,R}$

$$g^q(|u|) = g^q(|u-k|+k) \leq \frac{1}{2}g^q(2|u-k|)+\frac{1}{2}g^q(2k) \leq 2^{\mu q-1}\left\{g^q(|u-k|)+g^q(d)\right\},$$

we obtain

$$f(x,u,\mu q\frac{k-u}{\eta}D\eta) \leq \frac{c}{(t-s)^{\mu q}\eta^{\mu q}}\{g^q(|u-k|)+g^q(d)\}.$$

So, we get

$$f\left(x,(1-\eta^{\mu q})u+\eta^{\mu q}k,(1-\eta^{\mu q})Du+\eta^{\mu q}\mu q\frac{k-u}{\eta}D\eta\right)$$
$$\leq (1-\eta^{\mu q})^2 f(x,u,Du)+\frac{c}{(t-s)^{\mu q}}\{g^q(|u-k|)+g^q(d)\}.$$

Therefore, estimate (4.4) follows.

Conclusion.

By (4.4), adding to both sides Q times the left hand side, we get:

$$\int_{A_{k,s}} f(x,u,Du)\,dx \leq \frac{Q}{Q+1}\int_{A_{k,t}} f(x,u,Du)\,dx+\frac{c_3}{(t-s)^{\mu q}}\int_{A_{k,R}}\{g^q(u-k)+g^q(d)\}\,dx.$$

Thus, by Lemma 4.3 with $\tau_0=\rho$ and $\tau_1=R$ and

$$\phi(t)=\int_{A_{k,t}} f(x,u,Du)\,dx, \qquad A=\int_{A_{k,R}}\{g^q(u-k)+g^q(d)\}\,dx,$$

we get (4.1). □

5. Proof of Theorems 2.2, 2.3 and 2.6

We will use the following classical result, see e.g. [19].

Lemma 5.1. *Let $\alpha>0$ and (J_h) a sequence of real positive numbers, such that*

$$J_{h+1}\leq A\,\lambda^h J_h^{1+\alpha},$$

with $A>0$ and $\lambda>1$. If $J_0\leq A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha^2}}$, then $J_h\leq\lambda^{-\frac{h}{\alpha}}J_0$ and $\lim_{h\to\infty}J_h=0$.

We now need to introduce some notations.

Fixed $B_{R_0}(x_0)\Subset\Omega$, with $R\leq R_0$, define the decreasing sequences

$$\rho_h:=\frac{R}{2}+\frac{R}{2^{h+1}}=\frac{R}{2}(1+\frac{1}{2^h}), \qquad \bar\rho_h:=\frac{\rho_h+\rho_{h+1}}{2}=\frac{R}{2}(1+\frac{3}{4\cdot 2^h}).$$

Fixed a positive constant $d\geq t_0$, to be chosen later, define the increasing sequence of positive real numbers

$$k_h:=d\left(1-\frac{1}{2^{h+1}}\right),\ h\in\mathbb{N}.$$

Moreover, whenever $g(|u|)\in L^q_{\mathrm{loc}}(\Omega)$, define the sequence (J_h),

$$J_h:=\int_{A_{k_h,\rho_h}} g^q(u-k_h)\,dx.$$

We begin proving an inequality that will be the common root to prove Theorems 2.2 and 2.3.

Lemma 5.2. *Assume (H1) and (H2). Let $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$. Assume that $q<\overline{p}^*$ or, if $q=\overline{p}^*$, that $g(|u|)\in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$.*
If $2^hJ_h\geq 1$ for all h, then there exists a constant $C>0$ such that for all $h\in\mathbb{N}\cup\{0\}$

$$J_{h+1}\leq\frac{C}{(g(d))^{q-\frac{q^2}{\overline{p}^*}}}\left(\frac{1}{R}\right)^{\mu\frac{q^2}{p}}\lambda^h J_h^{1+\alpha},$$

where $\lambda=4^{\mu\frac{q^2}{p}}$ and $\alpha=\frac{q}{p}-\frac{q}{\overline{p}^*}$.

Proof. Since u is quasi-minimizer of $\mathcal{F}$ and (H2) holds, then $g(|u_{x_i}|) \in L^{p_i}_{\rm loc}(\Omega)$.
If $q < \overline{p}^*$, then $\max\{p_i\} < \overline{p}^*$ and, by Proposition 3.3, $g(|u|) \in L^{\overline{p}^*}_{\rm loc}(\Omega)$. If $q = \overline{p}^*$, we have, by assumption, that $g(|u|) \in L^{\overline{p}^*}_{\rm loc}(\Omega)$. In particular, $g(|u|) \in L^{p_i}_{\rm loc}(\Omega)$, $i = 1, ..., n$, and J_h is finite. Moreover, $J_{h+1} \leq J_h$, since the following chain of inequalities holds:

$$J_{h+1} \leq \int_{A_{k_{h+1},\rho_h}} g^q(u - k_{h+1})\, dx \leq \int_{A_{k_{h+1},\rho_h}} g^q(u - k_h)\, dx \leq J_h. \tag{5.1}$$

Let now define a sequence (ζ_h) of cut-off functions such that $\zeta_h \in C^\infty_c(B_{\bar\rho_h}(x_0))$, $\zeta_h \equiv 1$ in $B_{\rho_{h+1}}$, $|D\zeta_h| \leq \frac{2^{h+4}}{R}$.

By the Hölder inequality, denoting $(u - k_{h+1})_+ = \max\{u - k_{h+1}, 0\}$ we get

$$\begin{aligned} J_{h+1} &\leq |A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{A_{k_{h+1},\bar\rho_h}} (g(u - k_{h+1})\zeta_h)^{\overline{p}^*}\, dx \right)^{\frac{q}{\overline{p}^*}} \\ &= |A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{B_{\bar\rho_h}} (\zeta_h g((u - k_{h+1})_+))^{\overline{p}^*}\, dx \right)^{\frac{q}{\overline{p}^*}}. \end{aligned}$$

To apply the Sobolev embedding Theorem 3.1 to the function $g((u - k_{h+1})_+)\zeta_h$, we need to prove that $g((u - k_{h+1})_+)\zeta_h \in W^{1,(p_1,...,p_n)}_0(B_{\bar\rho_h}(x_0))$. Precisely, it remains only to prove that $(\zeta_h g((u - k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar\rho_h}(x_0))$. By Lemma 4.1 and using $(g((u(x) - k_{h+1})_+)))_{x_i} = g'(u(x) - k_{h+1})u_{x_i}(x)\chi_{A_{k_{h+1},\bar\rho_h}}(x)$, for a.e. $x \in B_{\bar\rho_h}(x_0)$, we get that for a.e. $x \in B_{\bar\rho_h}(x_0)$

$$\begin{aligned} &|(\zeta_h g((u - k_{h+1})_+))_{x_i}| \\ &\leq c(\mu)\frac{2^h}{R}\{g(u - k_{h+1}) + g(t_0)\}\chi_{A_{k_{h+1},\bar\rho_h}}(x) + \mu g(|u_{x_i}|)\chi_{A_{k_{h+1},\bar\rho_h}}(x). \end{aligned} \tag{5.2}$$

Indeed,

$$\begin{aligned} |(\zeta_h g((u - k_{h+1})_+))_{x_i}| &\leq g((u - k_{h+1})_+)|(\zeta_h)_{x_i}| + \zeta_h g'(u - k_{h+1})|u_{x_i}|\chi_{A_{k_{h+1},\bar\rho_h}}(x) \\ &\leq g((u - k_{h+1})_+)|D\zeta_h| + \zeta_h \left\{g'(u - k_{h+1})(u - k_{h+1}) + g'(|u_{x_i}|)|u_{x_i}|\right\}\chi_{A_{k_{h+1},\bar\rho_h}}(x) \\ &\leq g(u - k_{h+1})|D\zeta_h|\chi_{A_{k_{h+1},\bar\rho_h}}(x) + \zeta_h\mu\{g(u - k_{h+1}) + g(|u_{x_i}|) + 2g(t_0)\}\chi_{A_{k_{h+1},\bar\rho_h}}(x) \end{aligned}$$

and the claim follows. Since both $g(|u|)$ and $g(|u_{x_i}|)$ are in $L^{p_i}_{\rm loc}(\Omega)$, we have proved that $(\zeta_h g((u - k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar\rho_h}(x_0))$.
Thus, by the Sobolev embedding Theorem 3.1

$$J_{h+1} \leq c\,|A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\overline{p}^*}} \left\{ \sum_{i=1}^n \left(\int_{B_{\bar\rho_h}} |(\zeta_h g((u - k_{h+1})_+)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \right\}^q. \tag{5.3}$$

By (5.2), since $(a+b)^{\frac{1}{p_i}} \leq a^{\frac{1}{p_i}} + b^{\frac{1}{p_i}}$, $g(d) \geq g(t_0) \geq 1$, and $\bar\rho_h \leq \rho_h$, we get

$$\begin{aligned} &\left(\int_{B_{\bar\rho_h}} |(\zeta_h g((u - k_{h+1})_+)_{x_i}|^{p_i}\, dx \right)^{\frac{1}{p_i}} \\ &\leq \frac{c2^h}{R} \left(\int_{A_{k_{h+1},\rho_h}} \{g^q(u - k_{h+1}) + g^q(d)\}\, dx \right)^{\frac{1}{p_i}} + \mu \left(\int_{A_{k_{h+1},\bar\rho_h}} [g(|u_{x_i}|)]^{p_i}\, dx \right)^{\frac{1}{p_i}}. \end{aligned}$$

By (2.2) and the Caccioppoli inequality (4.1) we obtain

$$\begin{aligned} c_1 \int_{A_{k_{h+1},\bar\rho_h}} g^{p_i}(|u_{x_i}|)\,dx &\le \int_{A_{k_{h+1},\bar\rho_h}} f(x,u,Du)\,dx \\ &\le c\left(\frac{2^h}{R}\right)^{\mu q} \int_{A_{k_{h+1},\rho_h}} \{g^q(u-k_{h+1}) + g^q(d)\}\,dx, \end{aligned}$$

with c possibly depending on $\operatorname{diam}\Omega$. Collecting the above inequalities, we have

$$\left(\int_{A_{k_{h+1},\bar\rho_h}} |(\zeta_h g(u-k_{h+1}))_{x_i}|^{p_i}\,dx\right)^{\frac{1}{p_i}} \le c\left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}} \left(\int_{A_{k_{h+1},\rho_h}} \{g^q(u-k_{h+1}) + g^q(d)\}\,dx\right)^{\frac{1}{p_i}}.$$

By the above inequality, (5.3) and (5.1) it follows that

$$J_{h+1} \le c|A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\bar p^*}} \left\{\sum_{i=1}^n \left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}} \left(J_h \,+\, g^q(d)|A_{k_{h+1},\rho_h}|\right)^{\frac{1}{p_i}}\right\}^q. \tag{5.4}$$

Notice that

$$\begin{aligned} J_h &\ge \int_{A_{k_{h+1},\rho_h}} g^q(u-k_h)\,dx \ge g^q(k_{h+1}-k_h)|A_{k_{h+1},\rho_h}| \\ &= g^q(\frac{d}{2^{h+2}})|A_{k_{h+1},\rho_h}| \ge \frac{g^q(d)}{2^{(h+2)\mu q}}|A_{k_{h+1},\rho_h}|, \end{aligned}$$

therefore

$$|A_{k_{h+1},\bar\rho_h}| \le |A_{k_{h+1},\rho_h}| \le \frac{2^{(h+2)\mu q}}{g^q(d)} J_h. \tag{5.5}$$

Since $2^h J_h \ge 1$ for all h, by (5.1), (5.4), (5.5) and denoting $p = \min\{p_i\}$ we obtain

$$\begin{aligned} J_{h+1} &\le c\left(\frac{2^{h\mu q}}{g^q(d)} J_h\right)^{1-\frac{q}{\bar p^*}} \left\{\sum_{i=1}^n \left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}} \left(2^{h\mu q} J_h\right)^{\frac{1}{p_i}}\right\}^q \\ &\le c\left(\frac{2^{h\mu q}}{g^q(d)} J_h\right)^{1-\frac{q}{\bar p^*}} \left(\frac{2^h}{R}\right)^{\mu\frac{q^2}{p}} \left(2^{h\mu q} J_h\right)^{\frac{q}{p}} \le \frac{C}{R^{\mu\frac{q^2}{p}}\,(g^q(d))^{\frac{\bar p^*-q}{\bar p^*}}} \left(4^{\mu\frac{q^2}{p}}\right)^h J_h^{1+\frac{q}{p}-\frac{q}{\bar p^*}} \end{aligned}$$

and the conclusion follows. □

We are now ready to prove the first of our main results.

Proof of Theorem 2.2. Let us assume that $2^h J_h \ge 1$ for all h and let d be a positive constant, $d \ge t_0$, to be chosen later.
By Lemma 5.2 we have that for all h

$$J_{h+1} \le \frac{C}{(g(d))^{q-\frac{q^2}{\bar p^*}}} \left(\frac{1}{R}\right)^{\mu\frac{q^2}{p}} \lambda^h J_h^{1+\alpha},$$

with $\lambda = 4^{\mu\frac{q^2}{p}}$ and $\alpha = \frac{q}{p} - \frac{q}{\bar p^*} > 0$.

Using Lemma 5.1 with $A = \dfrac{C}{R^{\mu \frac{q^2}{p}} (g^q(d))^{\frac{\overline{p}^*-q}{\overline{p}^*}}}$, we have that if

$$J_0 \le K[g(d)]^{p\frac{\overline{p}^*-q}{\overline{p}^*-p}}, \qquad \text{with } \ K := \left\{ \frac{C}{R^{\mu \frac{q^2}{p}}} \right\}^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}, \tag{5.6}$$

then $\lim_{h\to+\infty} J_h = 0$.
Since

$$J_0 := \int_{A_{\frac{d}{2},R}} g^q(u - \frac{d}{2})\, dx \le \int_{B_R} g^q(|u|)\, dx,$$

it is easy to check that (5.6) is satisfied if we choose d such that

$$g(d) = g(t_0) + \left\{ \frac{1}{K} \int_{B_R} g^q(|u|)\, dx \right\}^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}}. \tag{5.7}$$

Hence, since $\lim\limits_{h\to+\infty} J_h = \int_{A_{d,\frac{R}{2}}} g^q(u-d)\, dx$, we get $|A_{d,\frac{R}{2}}| = 0$. So, we conclude that $B_{\frac{R}{2}} \subseteq \{u \le d\}$.

On the other hand, since $-u$ is a quasi-minimizer of the functional

$$\mathcal{I}(v) = \int \overline{f}(x, u, Du)\, dx,$$

where $\overline{f}(x, u, \xi) = f(x, -u, -\xi)$, which satisfies the same assumptions of f, we obtain that $B_{\frac{R}{2}} \subseteq \{u \ge -d\}$.
Therefore, by (5.7) and the monotonicity of g,

$$g(|u|) \le g(t_0) + \left\{ \left(\frac{C}{R^{\mu \frac{q^2}{p}}} \right)^{\frac{1}{\alpha}} \lambda^{\frac{1}{\alpha^2}} \int_{B_R} g^q(|u|)\, dx \right\}^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}} \qquad \text{a.e. in} \quad B_{\frac{R}{2}},$$

that is

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le g(t_0) + \frac{c}{R^{\mu \frac{q\overline{p}^*}{p(\overline{p}^*-q)}}} \left(\int_{B_R} g^q(|u|)\, dx \right)^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}}.$$

The estimate (2.6) follows.

Now, let us assume that it fails that $2^h J_h \ge 1$ for all h. Then, for a suitable subsequence $J_{h_m} \to 0$, hence $g^q(u - d) = 0$ a.e. in $A_{d,\frac{R}{2}}$, for any $d \ge t_0$. Choosing d as in (5.7) we get the same estimate than in the previous case. □

We now turn to the proof of our boundedness result under the assumption $q = \overline{p}^*$.

Proof of Theorem 2.3. As in the proof of Theorem 2.2, we observe that without loss of generality we may suppose that $2^h J_h \ge 1$ for all h.

If $\max\{p_i\} = \overline{p}^*$, we know, by assumption, that $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. The same conclusion holds if $\max\{p_i\} < \overline{p}^*$. Indeed, (H2) implies $g(|u_{x_i}|) \in L^{p_i}_{\mathrm{loc}}(\Omega)$, so, by Proposition 3.3, $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$.
By Lemma 5.2,

$$J_{h+1} \le C \left(\frac{1}{R} \right)^{\mu \frac{(\overline{p}^*)^2}{p}} \lambda^h J_h^{1+\alpha},$$

with $\lambda = 4^{\mu\frac{(\overline{p}^*)^2}{p}}$ and $\alpha = \frac{\overline{p}^*}{p} - 1 > 0$. Therefore, by Lemma 5.1 we have that $\lim_{h\to+\infty} J_h = 0$, if

$$J_0 \leq \left(C \left(\frac{1}{R} \right)^{\mu\frac{(\overline{p}^*)^2}{p}} \right)^{-\frac{1}{\alpha}} \left(4^{\mu\frac{(\overline{p}^*)^2}{p}} \right)^{-\frac{1}{\alpha^2}} . \tag{5.8}$$

By definition, $J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(u - \frac{d}{2})\, dx$. Thus, we choose d large, such that (5.8) holds. This is possible, since $g^{\overline{p}^*}(|u|) \in L^1(B_R)$, then

$$J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(u - \frac{d}{2})\, dx \leq \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(|u|)\, dx \to_{d\to+\infty} 0.$$

Hence, there exists $d > 0$ such that (5.8) hold and with such a choice of d we get $J_h \longrightarrow 0$, i.e.

$$\int_{A_{d,\frac{R}{2}}} g^{\overline{p}^*}(u - d)\, dx = 0.$$

Therefore, $u \leq d$ a.e. in $B_{\frac{R}{2}}(x_0)$.
To get a bound from below, we proceed as in the proof of Theorem 2.2. □

We conclude the section with the proof of Theorem 2.6.

Proof of Theorem 2.6. If $q < \overline{p}^*$ then we get the thesis by Theorem 2.2. Assume $q = \overline{p}^*$. By $\mathcal{F}(u_0) < +\infty$ and (H3), we get $u_0 \in W^{1,(p_1,\dots,p_n)}(\Omega)$. Theorem 3.1 implies $u - u_0 \in L^{\overline{p}^*}(\Omega)$. Thus, $u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. The conclusion follows by Theorem 2.3. □

6. Proof of Theorem 2.7

In this section, we assume the growth condition (H3) with $\min\{p_i\} = 1$. For the reader's convenience, we now recall the main notations. The functional is

$$\mathcal{F}(u) = \int_\Omega f(x, u, Du)\, dx,$$

and we assume that there exist $c_1, c_2 > 0$ and $1 = \min\{p_i\} \leq p_i \leq q$, $i = 1, \dots, n$, such that

$$c_1 \sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x, s, \xi) \leq c_2 \{1 + |\xi|^q + |s|^q\} \tag{6.1}$$

for a.e. x and every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

Fixed $u_0 \in W^{1,1}(\Omega)$ such that $\mathcal{F}(u_0) < +\infty$, we consider the relaxed functional in $BV(\Omega)$ of $\mathcal{F}$

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k\to+\infty} \mathcal{F}(u_k) \,:\, u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega) \right\}.$$

Proof of Theorem 2.7. By the Rellich's Theorem in BV, every minimizing sequence for $\mathcal{F}$ in $u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$ has a L^1-convergent subsequence. The lower semicontinuity of $\overline{\mathcal{F}}$ gives the existence of a minimizer $\bar{u}$ in BV, such that

$$\overline{\mathcal{F}}(\bar{u}) = \min_{u\in BV} \overline{\mathcal{F}}(u) = \inf_{u\in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)} \mathcal{F}(u). \tag{6.2}$$

We prove now that $\bar{u}$ is locally bounded. By the minimality of $\bar{u}$ and (6.2) there exists a sequence (u_k) in $u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$ such that for all k

$$\mathcal{F}(u_k) \le \inf_{u_0+W_0^{1,(p_1,\dots,p_n)}(\Omega)} \mathcal{F} + \frac{1}{k}, \qquad \text{and} \qquad u_k \to_{k\to+\infty} \bar{u} \text{ in } L^1(\Omega). \tag{6.3}$$

By the Ekeland's variational principle, see [14], for every k there exists $v_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$ such that

$$\mathcal{F}(v_k) \le \mathcal{F}(u) + \frac{1}{\sqrt{k}} \sum_{i=1}^n \left(\int_\Omega |(v_k - u)_{x_i}|^{p_i} \, dx \right)^{1/p_i} \qquad \forall u \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega) \tag{6.4}$$

and

$$\sum_{i=1}^n \left(\int_\Omega |(v_k - u_k)_{x_i}|^{p_i} \, dx \right)^{1/p_i} \le \frac{1}{\sqrt{k}} \quad \forall k. \tag{6.5}$$

Since $u_k - v_k \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$, then (6.5) implies that $u_k - v_k$ converges to 0 in L^1. Thus, by the second item in (6.3), we get that $v_k \to \bar{u}$ in L^1.
Notice that there exists $\tilde{c} > 0$, depending on $|\Omega|$, such that

$$a^{1/p_i} \le a + \tilde{c}|\Omega| \qquad \forall\ i = 1, \dots, n, \text{ and } \forall\ a > 0.$$

Thus, using (6.4) and (H2) we get that for all $u \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$

$$\begin{aligned} \mathcal{F}(v_k) &\le \mathcal{F}(u) + \frac{1}{\sqrt{k}} \left\{ \sum_{i=1}^n \left(\int_\Omega |(v_k)_{x_i}|^{p_i} \, dx \right)^{1/p_i} + \sum_{i=1}^n \left(\int_\Omega |u_{x_i}|^{p_i} \, dx \right)^{1/p_i} \right\} \\ &\le \left(1 + \frac{1}{c_1\sqrt{k}} \right) \mathcal{F}(u) + \frac{1}{c_1\sqrt{k}} \mathcal{F}(v_k) + \frac{2\tilde{c}|\Omega|}{\sqrt{k}}, \end{aligned}$$

that implies

$$\left(1 - \frac{1}{c_1\sqrt{k}} \right) \mathcal{F}(v_k) \le \left(1 + \frac{1}{c_1\sqrt{k}} \right) \mathcal{F}(u) + \frac{2\tilde{c}|\Omega|}{\sqrt{k}}.$$

Therefore, the above inequality implies that v_k is a quasi-minimizer of the functional

$$\mathcal{I}(u) := \int_\Omega (f(x, u, Du) + 1) \, dx,$$

with Q independent of k.

Since $(x, s, \xi) \mapsto f(x, s, \xi) + 1$ satisfies properties analogous to (H1) and (6.1), we can apply Theorem 2.2. Thus, $v_k \in L^\infty_{\text{loc}}(\Omega)$ and it satisfies an estimate analogous to (2.6), that we now write using cubes instead than balls. Precisely, fixed $x_0 \in \Omega$, consider $Q_R(x_0) \Subset \Omega$, cube centered at x_0 with edges of lenght $2R$ parallel to the coordinate axes. Then there exists a constant c, independent of k, such that

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c \left\{ 1 + \frac{1}{R^{\mu \frac{q\bar{p}^*}{\bar{p}^*-q}}} \left(\int_{Q_R(x_0)} |v_k|^q \, dx \right)^{\frac{1+\theta}{q}} \right\}, \tag{6.6}$$

where $\theta = \frac{\bar{p}^*(q-1)}{\bar{p}^*-q}$.
Since $\mathcal{F}(u_0) < +\infty$, then $u_0 \in W^{1,(p_1,\dots,p_n)}(\Omega)$. By Theorem 3.2, $u_0 \in L^{\bar{p}^*}(Q_R)$ and it satisfies an estimate as in (3.1) on the cube Q_R. Moreover, since $v_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$, we can apply

Theorem 3.1 to the function $v_k - u_0 \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$. Thus,

$$\left\{\int_{Q_R(x_0)} |v_k|^q\,dx\right\}^{\frac{1}{q}} \le |\Omega|^{1-\frac{q}{\overline{p}^*}} \left\{\left(\int_\Omega |v_k-u_0|^{\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}} + \left(\int_{Q_R(x_0)} |u_0|^{\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}}\right\}$$

$$\le c(\Omega)\sum_{i=1}^n \left(\int_\Omega |(v_k-u_0)_{x_i}|^{p_i}\,dx\right)^{\frac{1}{p_i}} + c(\Omega)\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}.$$

Using (6.1) it is easy to prove that there exists $c>0$ independent of k, such that

$$\sum_{i=1}^n \left(\int_\Omega |(v_k-u_0)_{x_i}|^{p_i}\,dx\right)^{\frac{1}{p_i}} \le c\left\{\mathcal{F}(v_k)+1\right\} + c\sum_{i=1}^n \left\{\int_\Omega |(u_0)_{x_i}|^{p_i}\,dx\right\}^{\frac{1}{p_i}}$$
$$\le c\left\{\mathcal{F}(v_k)+1\right\} + c\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}. \tag{6.7}$$

Thus, collecting (6.6)-(6.7) we get

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\mathcal{F}(v_k)+1+\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{1+\theta}\right\}, \tag{6.8}$$

for some positive c depending also on Ω but independent of k and u_0.
By (6.4) applied with u_k in place of u, by (6.5) and the first property in (6.3), we have

$$\mathcal{F}(v_k) \le \mathcal{F}(u_k) + \frac{1}{k} \le \inf_{u_0+W_0^{1,(p_1,\dots,p_n)}(\Omega)} \mathcal{F} + \frac{2}{k}.$$

Therefore, (6.8) implies

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\inf_{u_0+W_0^{1,(p_1,\dots,p_n)}(\Omega)} \mathcal{F} + \frac{2}{k} + 1 + \|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{1+\theta}\right\}.$$

So, up to subsequences, v_k converges to a function v in the $*$-weak topology of L^∞. Since v_k also converges to $\bar{u}$ in L^1, then $v=\bar{u}$. By the lower semicontinuity of the L^∞-norm and (6.2), we conclude. □

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